Joint Physics Analysis Center

Summer Workshop on the Reaction Theory
Exercise sheet 4

Team 2: Andrew Jackura and Marc Vanderhaeghen

Contact: http://www.indiana.edu/~ssrt/index.html

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Classwork

Elastic Unitarity for Partial Waves

Consider the $2 \to 2$ elastic scattering of scalar particles a and b. We consider an energy range where only the two particles a and b can be produced, nothing more. The amplitude for this process is denoted $\mathcal{A}(s, \widehat{\mathbf{p}} \cdot \widehat{\mathbf{p}}')$, where $s = (p_a + p_b)^2$, and $\widehat{\mathbf{p}} = (\theta, \varphi)$ is the orientation of the initial a in the center-of-momentum system (CMS), and $\widehat{\mathbf{p}}' = (\theta', \varphi')$ is the orientation of the final a in the CMS. Note that $\cos\Theta = \widehat{\mathbf{p}} \cdot \widehat{\mathbf{p}}'$, where Θ is the scattering angle in the CMS (see Fig. 1).

(1) The elastic unitarity equation for the amplitude (see Fig. 2)

$$\text{Im}\,\mathcal{A}(s, \widehat{\mathbf{p}} \cdot \widehat{\mathbf{p}}') = \rho(s) \int \frac{d\widehat{\mathbf{p}}''}{4\pi} \mathcal{A}(s, \widehat{\mathbf{p}}' \cdot \widehat{\mathbf{p}}'') \mathcal{A}(s, \widehat{\mathbf{p}} \cdot \widehat{\mathbf{p}}''), \tag{1}$$

where

$$\rho(s) = \frac{1}{2} \frac{1}{8\pi} \frac{2|\mathbf{p}''|}{\sqrt{s}}, \tag{2}$$

and $|\mathbf{p}''| = \lambda^{1/2}(s, m_a^2, m_b^2)/(2\sqrt{s})$ is the momentum of the intermediate state, and the orientation of particle a in the intermediate state is $\widehat{\mathbf{p}}''$. Verify the elastic unitarity relation for partial wave amplitudes,

$$\text{Im}\, a_\ell(s) = \rho(s) |a_\ell(s)|^2, \tag{3}$$

where the partial waves are defined by the expansion

$$\mathcal{A}(s, \widehat{\mathbf{p}} \cdot \widehat{\mathbf{p}}') = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(s) P_\ell(\widehat{\mathbf{p}} \cdot \widehat{\mathbf{p}}'). \tag{4}$$

One can write the unitarity relation for the inverse partial wave amplitude $a_\ell^{-1}(s)$. Derive the elastic unitarity relation for the inverse partial wave amplitude $a_\ell^{-1}(s)$.

(2) Show that elastic partial wave amplitudes can be written in terms of one function, the phase shift $\delta_\ell(s)$, in the form

$$a_\ell(s) = \frac{e^{i\delta_\ell(s)} \sin\delta_\ell(s)}{\rho(s)}. \tag{5}$$

(**Hint**: The complex partial wave amplitude can be written as $a_\ell(s) = |a_\ell(s)| e^{i\delta_\ell(s)}$. Substitute this into Eq. (3).)

(3) Show that the real and imaginary parts satisfy the equation of a circle. Where is the center of the circle? What is the radius of the circle?

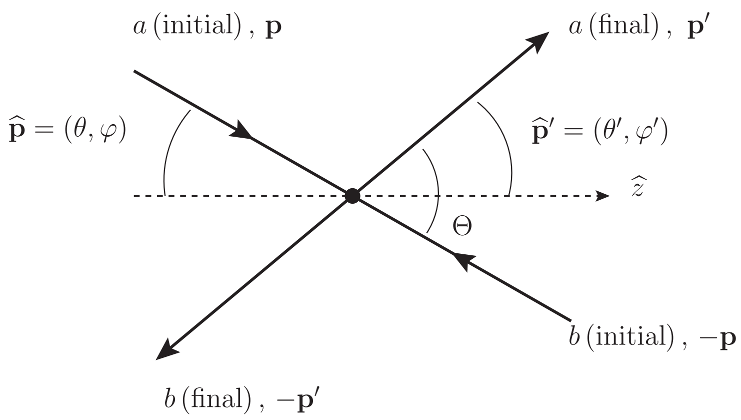

Figure 1: Center-of-momentum system (CMS) elastic scattering of $ab \to ab$. The scattering angle Θ (between initial a and final a) in terms of the initial and final orientaions is $\cos\Theta = \widehat{\mathbf{p}} \cdot \widehat{\mathbf{p}}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$.

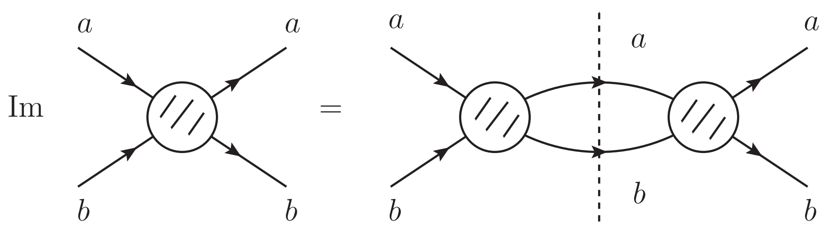

Figure 2: Diagram representing the elastic unitarity equation (1).

Källén Triangle function

The Källén triangle function, denoted $\lambda(x, y, z)$, is totally symmetric in the arguments, has the form

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz), \tag{6}$$

$$= \left[x - (\sqrt{y} + \sqrt{z})^2\right]\left[x - (\sqrt{y} - \sqrt{z})^2\right], \tag{7}$$

$$= (x - y - z)^2 - 4yz. \tag{8}$$

Some special cases:

$$\lambda(x, y, y) = x(x - 4y) \tag{9}$$

$$\lambda(x, y, 0) = (x - y)^2 \tag{10}$$

$$\lambda(x, 0, 0) = x^2 \tag{11}$$

Properties of Legendre functions P_ℓ.

Orthogonality relation

$$\int d\widehat{\mathbf{p}}''\, P_\ell(\widehat{\mathbf{p}}' \cdot \widehat{\mathbf{p}}'') P_{\ell'}(\widehat{\mathbf{p}} \cdot \widehat{\mathbf{p}}'') = \frac{4\pi}{2\ell + 1} \delta_{\ell'\ell}\, P_\ell(\widehat{\mathbf{p}} \cdot \widehat{\mathbf{p}}') \tag{12}$$