Joint Physics Analysis Center

Summer Workshop on the Reaction Theory
Exercise sheet 3

Team 2: Andrew Jackura and Marc Vanderhaeghen

Contact: http://www.indiana.edu/~ssrt/index.html

June 12 – June 22 To be discussed on Wednesday of Week-I.

Classwork

Partial Waves of the Born Amplitude

Consider the $2 \to 2$ process $aa \to bb$, with $m_a < m_b$, where the a and b particles are pseudoscalars ($J^P = 0^-$). The process includes a t-channel exchange of particle c, which is a scalar meson ($J^P = 0^+$), with mass M. The amplitude is of the form

$$\mathcal{A}(s,t) = \frac{g^2}{t - M^2}, \tag{1}$$

where g is the coupling between the a, b, and c particles (see Fig. 1). Work in the center-of-momentum system (CMS).

3.1 s-channel

(1) Write t in terms of the masses, s, and $z_s = \cos\theta_s$, where θ_s is the scattering angle in the s-channel CMS.

(2) Write the allowed J^P quantum numbers of the (bb) two-particle system.

(3) The partial wave expansion is defined as

$$\mathcal{A}(s,t) = \sum_{\ell=0}^{\infty}(2\ell+1)\, a_\ell(s) P_\ell(\cos\theta_s), \tag{2}$$

where P_ℓ are the Legendre functions of the 1^{st} kind. Write the S-wave partial wave amplitude of the Born amplitude Eq. (1).

(4) Write the partial wave amplitudes $a_\ell(s)$ of the Born amplitude Eq. (1) in terms of the Legendre functions of the 2^{nd} kind,

$$Q_\ell(z) = \frac{1}{2}\int_{-1}^{+1} dz' \frac{P_\ell(z')}{z - z'}. \tag{3}$$

(5) For $s \to s_{th}^b \equiv 4m_b^2$, what is the behavior of $a_\ell(s)$?

(6) Where do the branch cuts occur for the $\ell = 0$ partial wave? Draw the branch cuts for $a_{\ell=0}(s)$ in the complex s-plane.

3.2 t-channel

Now consider the t-channel process of the above reaction ($a\bar{b} \to \bar{a}b$). The t-channel partial wave expansion is

$$\mathcal{A}(s,t) = \sum_{L=0}^{\infty}(2L+1)\, a_L(t) P_L(\cos\theta_t), \tag{4}$$

where θ_t is the scattering angle in the t-channel CMS. Find the t-channel partial wave amplitudes $a_L(t)$ for the Born amplitude Eq. (1).

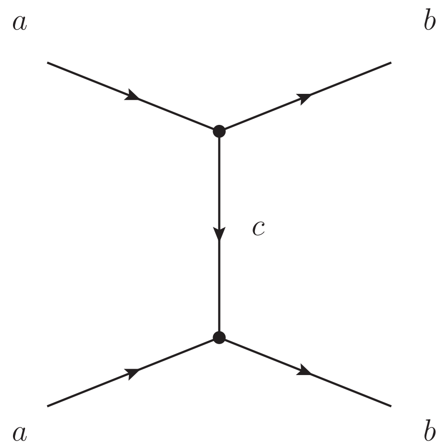

Figure 1: Diagram for Born amplitude in (1).

Properties of P_ℓ

In the following we consider only ℓ integer. The first few P_ℓ functions are (see Fig. 2)

$$P_0(z) = 1 \tag{5}$$
$$P_1(z) = z \tag{6}$$
$$P_2(z) = \frac{1}{2}(3z^2 - 1). \tag{7}$$

In general, all $P_\ell(z)$ can be found by the recursion relation

$$P_0(z) = 1\ , \quad P_1(z) = z \tag{8}$$
$$P_\ell(z) = z(2-\ell^{-1})P_{\ell-1}(z) - (1-\ell^{-1})P_{\ell-2}(z) \text{ for } \ell > 1. \tag{9}$$

The functions are orthogonal

$$\int_{-1}^{+1} dz\, P_{\ell'}(z) P_\ell(z) = \frac{2}{2\ell+1}\delta_{\ell'\ell}. \tag{10}$$

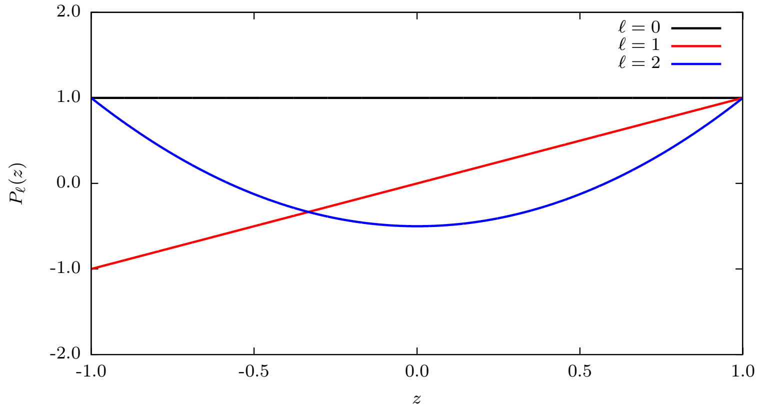

Figure 2: P_ℓ for $\ell = 0, 1, 2$.

Properties of Q_ℓ

In the following we consider only ℓ integer. The first few Q_ℓ functions are (see Fig. 3)

$$Q_0(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \tag{11}$$
$$Q_1(z) = \frac{z}{2}\ln\left(\frac{1+z}{1-z}\right) - 1 \tag{12}$$
$$Q_2(z) = \frac{3z^2-1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{3z}{2}. \tag{13}$$

In general, all $Q_\ell(z)$ can be found by the recursion relation

$$Q_0(z) = \frac{1}{2}\ln\frac{z+1}{z-1}\ , \quad Q_1(z) = zQ_0(z) - 1 \tag{14}$$
$$Q_\ell(z) = z(2-\ell^{-1})Q_{\ell-1}(z) - (1-\ell^{-1})Q_{\ell-2}(z) \text{ for } \ell > 1. \tag{15}$$

The asymptotic behavior of the Q_ℓ function is

$$Q_\ell(z) \to \frac{\pi^{1/2}}{(2z)^{\ell+1}}\frac{\ell!}{(\ell+\frac{1}{2})!} \qquad \text{as } z \to \infty. \tag{16}$$

Note the completeness condition

$$\sum_{\ell=0}^{\infty}(2\ell+1)Q_\ell(z')P_\ell(z) = \frac{1}{z'-z}. \tag{17}$$

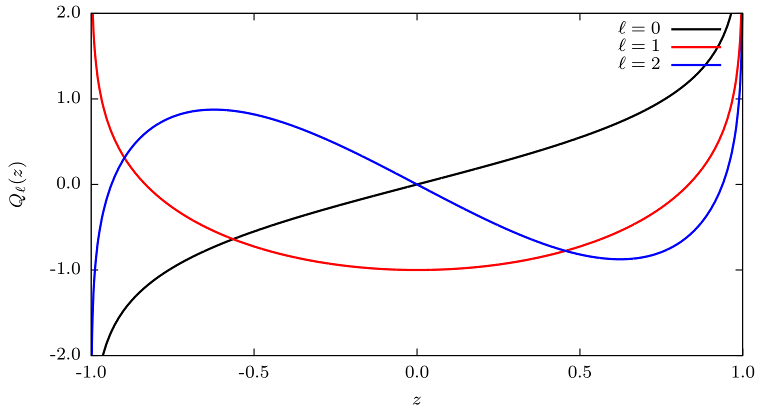

Figure 3: Q_ℓ for $\ell = 0, 1, 2$.