Joined Physics Analysis Center

Summer Workshop on the Reaction Theory
Exercise sheet 8

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Classwork

1. Derive all the quantum numbers $I^G J^{PC}$ in the $t-$channel of the following reactions
   (a) $\pi\pi \to \pi\pi$ and $K\bar{K} \to K\bar{K}$
   (b) $\pi N \to \pi N$, $\pi N \to \eta N$ and $KN \to KN$
   (c) $\gamma N \to \eta N$ and $\gamma N \to \pi N$
   (d) $\pi\rho \to \rho\pi$

   Notation: $\pi = (\pi^+, \pi^-, \pi^0)$; $\rho = (\rho^+, \rho^-, \rho^0)$; $K = (K^+, K^0)$; $N = (p, n)$.

2. Assume that the Regge exchange form a $SU(3)$ octet and a $SU(3)$ singlet with the coupling for the octet and the singlet being different. Consider a vector and a tensor nonet (octet plus singlet). From the duality hypothesis and the absence of double charge meson, find the combination of octet-singlet tensor that decouples from $\pi\pi$. Use the SU(3) Clebsch-Gordan coefficients from Rev.Mod.Phys. 36 (1964) 1005. What are the quark content and the $K\bar{K}$ couplings of these states?

3. Assuming ideal mixing for the vector and tensor, derive the exchange degeneracy relations coming duality and the absence of resonance in the following reactions
   (a) $\pi\pi \to \pi\pi$
   (b) $K\bar{K} \to K\bar{K}$
   (c) $KN \to KN$
   (d) $\pi\rho \to \rho\pi$ (and $\pi\pi \to \rho\rho$)

4. Derive a Lorentz-covariant basis, the isospin decomposition and the crossing properties for the following reactions
   (a) $\pi N \to \pi N$ and $KN \to KN$
   (b) $NN \to NN$
   (c) $\omega \to \pi\pi\pi$ and $B \to J/\psi K\pi$
   (d) $\pi\rho \to \pi\rho$
   (e) $\gamma N \to \pi N$ and $\gamma^* N \to \pi N$ (use $F^{\mu\nu} = \epsilon^\mu k^\nu - \epsilon^\nu k^\mu$)

Solution

1. The list of exchanges having only $I = 0, 1$ is presented on Table 1. Notation: signature $\tau = (-1)^J$ and naturality $\eta = P(-1)^J$. In the quark model, $P = (-1)^{\ell+1}$ and $C = (-1)^{\ell+S}$, hence 0^{--}, $(1, 3, 5, \ldots)^{-+}$ and $(0, 2, 4, \ldots)^{+-}$ are forbidden in the quark model. Let's refer to these quantum numbers as "exotic".

Table 1: Regge Trajectories

$I^{G\tau\eta}$		J^{PC}	$I^{G\tau\eta}$		J^{PC}
0^{+++}	f_+	$(0, 2, 4, \ldots)^{++}$	0^{+--}	f_-	$(1, 3, 5, \ldots)^{++}$
0^{--+}	ω_-	$(1, 3, 5, \ldots)^{--}$	0^{-+-}	ω_+	$(0, 2, 4, \ldots)^{--}$
1^{-++}	a_+	$(0, 2, 4, \ldots)^{++}$	1^{---}	a_-	$(1, 3, 5, \ldots)^{++}$
1^{+-+}	ρ_-	$(1, 3, 5, \ldots)^{--}$	1^{++-}	ρ_+	$(0, 2, 4, \ldots)^{--}$
0^{++-}	η_+	$(0, 2, 4, \ldots)^{-+}$	0^{+-+}	η_-	$(1, 3, 5, \ldots)^{-+}$
0^{---}	h_-	$(1, 3, 5, \ldots)^{+-}$	0^{-++}	h_+	$(0, 2, 4, \ldots)^{+-}$
1^{-+-}	π_+	$(0, 2, 4, \ldots)^{-+}$	1^{--+}	π_-	$(1, 3, 5, \ldots)^{-+}$
1^{+--}	b_-	$(1, 3, 5, \ldots)^{+-}$	1^{+++}	b_+	$(0, 2, 4, \ldots)^{+-}$

(a) for $\pi\pi$: $G = +$, $\eta = +$ and $\eta(-1)^I = +$ (Bose symmetry) $\Rightarrow f_+$ and ρ_-.
for $K\bar{K}$: $\eta = +$ and $\eta(-1)^I = + \Rightarrow f_+$, a_+, ω_- and ρ_-.

(b) for $\pi\eta$: $G = -$, $\eta = +$, $I = 1$; for NN: $I = 0, 1$ and no exotic $\Rightarrow a_+$.
for $K\bar{K}$: $\eta = +$; for NN: $I = 0, 1$ and no exotic $\Rightarrow f_+$, a_+, ω_- and ρ_-.

(c) for $\gamma\eta$ and $\gamma\pi^0$: $C = -$; for NN: $I = 0, 1$ and no exotic. $\Rightarrow \omega_\pm$, $\rho_\pm$, b_- and h_-.
for $\gamma\pi^+$: $I = 1$; for NN: $I = 0, 1$ and no exotic. $\Rightarrow a_\pm$, $\rho_\pm$, b_- and π_-.

(d) for $\pi\rho$: $G = - \Rightarrow a_\pm$, $\pi_\pm$, $\omega_\pm$ and $h_\pm$

Table 2: Exchanges

(a)	$\pi^+\pi^\mp \to \pi^+\pi^\mp$	$f_+ \pm \rho_-$
	$\pi^0\pi^0 \to \pi^0\pi^0$	f_+
	$K^+K^\mp \to K^+K^\mp$	$f_+ \pm \omega_- + a_+ \pm \rho_-$
	$K^+K^0 \to K^0K^+$	$a_+ - \rho_-$
(b)	$\pi^-p \to \eta n$	a_+
	$\pi^-p \to \pi^0 n$	$\sqrt{2}\rho_+$
	$\pi^\mp p \to \pi^\mp p$	$f_+ \pm \rho_+$
	$\pi^\mp n \to \pi^\mp n$	$f_+ \mp \rho_+$
	$K^-p \to \bar{K}^0 p$	$\sqrt{2}\,(\rho_- + a_+)$
	$K^+n \to K^0 p$	$\sqrt{2}\,(\rho_- - a_+)$
	$K^\mp p \to K^\mp p$	$f_+ \pm \rho_- + a_+ \pm \omega_-$
	$K^\mp n \to K^\mp n$	$f_+ \mp \rho_- - a_+ \pm \omega_-$
(c)	$\gamma p \to \eta p$	$(\omega_- + \rho_-) + (h_- + b_- + \omega_+ + \rho_+)$
	$\gamma p \to \pi^0 p$	$(\omega_- + \rho_-) + (h_- + b_- + \omega_+ + \rho_+)$
	$\gamma p \to \pi^+ n$	$(\rho_- + a_+) + (b_- + \pi_+ + \rho_+ + a_-)$
	$\gamma n \to \pi^- p$	$(\rho_- - a_+) + b_- - \pi_+ + \rho_+ - a_-)$
(d)	$\pi^+\rho^0 \to \rho^0\pi^+$	$(a_+ + \pi_-) + (a_- + \pi_+)$
	$\pi^+\rho^+ \to \rho^+\pi^+$	$(\omega_- - a_+ + h_+ - \pi_-) + (\omega_+ - a_- + h_- - \pi_+)$
	$\pi^+\rho^+ \to \pi^+\rho^+$	$f_+ - \rho_-$

2. For a general treatment of exchange degeneracy using group theory, see Ref. [1].

We assume that the residues obey a $SU(3)$ symmetry:

$$\beta_{ac}^{R}(t) \propto \langle \mathbf{8}\, Y_a\, I_a\, I_{a3}; \mathbf{8}\, Y_c^*\, I_c\, I_{c3}^* | \mathbf{8}\, Y_R\, I_R\, I_{R3} \rangle \,, \tag{1}$$

where $Y^* = -Y$ and $I_3^* = -I_3$. The hypercharge Y is the strangeness and I_3 is the isospin projection. The Clebsch-Gordan coefficients for $SU(3)$ are listed in Ref. [2]. Note the extra minus sign for the π^+ and K^-.

The four couplings are $\beta_{\mathbf{8}V}, \beta_{\mathbf{8}T}, \beta_{\mathbf{1}V}$ and $\beta_{\mathbf{1}T}$ for the octet/ singlet for the tensor and vector trajectories. The absence of isospin 2 meson in $\pi^+K^+ \to K^+\pi^+$ and in $\pi^+\pi^+ \to \pi^+\pi^+$ lead to

$$\pi^+K^+ \to K^+\pi^+ : \qquad \frac{3}{10}\beta_{\mathbf{8}T}^2 s^{\alpha_{\mathbf{8}T}} - \frac{1}{6}\beta_{\mathbf{8}V}^2 s^{\alpha_{\mathbf{8}V}} = 0 \tag{2a}$$

$$\pi^+\pi^+ \to \pi^+\pi^+ : \qquad \frac{1}{8}\beta_{\mathbf{1}T}^2 s^{\alpha_{\mathbf{1}T}} + \frac{1}{5}\beta_{\mathbf{8}T}^2 s^{\alpha_{\mathbf{8}T}} - \frac{1}{3}\beta_{\mathbf{8}V}^2 s^{\alpha_{\mathbf{8}T}} = 0 \tag{2b}$$

We combine them to get $\alpha_{\mathbf{8}V} = \alpha_{\mathbf{8}V} = \alpha_{\mathbf{1}T}$ and $(2/5)\beta_{\mathbf{8}T}^2 = (1/8)\beta_{\mathbf{1}T}^2$, I choose by convention

$$\sqrt{\frac{1}{8}}\beta_{\mathbf{1}T} = -\sqrt{\frac{2}{5}}\beta_{\mathbf{1}V}. \tag{3}$$

Let us define the octet-singlet mixing

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} \cos\theta_T & \sin\theta_T \\ -\sin\theta_T & \cos\theta_T \end{pmatrix} \begin{pmatrix} f_8 \\ f_1 \end{pmatrix} \tag{4}$$

The states are $f_8 = |\mathbf{8}; 000\rangle$ and $f_1 = |\mathbf{1}; 000\rangle$. The notation is $|\boldsymbol{R}; Y I I_3\rangle$. Let us impose that the f' coupling to $\pi^+\pi^+$ vanishes

$$-\sin\theta_T \left(-\sqrt{\frac{1}{5}}\beta_{\mathbf{8}T} \right) + \cos\theta_T \left(\sqrt{\frac{1}{8}}\beta_{\mathbf{1}T} \right) = 0. \tag{5}$$

With the relation between the couplings, we obtain $\sin\theta_T = \sqrt{2}\cos\theta_T$ or $\tan^2\theta_T = 1/2$. The quark content are then

$$\begin{pmatrix} \frac{u\bar{u}+d\bar{d}}{\sqrt{2}} \\ s\bar{s} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} \frac{u\bar{u}+d\bar{d}-2s\bar{s}}{\sqrt{6}} \\ \frac{u\bar{u}+d\bar{d}+s\bar{s}}{\sqrt{3}} \end{pmatrix} \tag{6}$$

The couplings are

$$\beta_{\pi\pi}^{f} = -\sqrt{\frac{1}{5}}\beta_{\mathbf{8}T}\cos\theta_T + \sqrt{\frac{1}{8}}\beta_{\mathbf{1}T}\sin\theta_T = -\sqrt{\frac{3}{5}}\beta_{\mathbf{8}T} \tag{7a}$$

$$\beta_{K^+K^+}^{f} = \sqrt{\frac{1}{20}}\beta_{\mathbf{8}T}\cos\theta_T + \sqrt{\frac{1}{8}}\beta_{\mathbf{1}T}\sin\theta_T = -\frac{1}{2}\sqrt{\frac{3}{5}}\beta_{\mathbf{8}T} \tag{7b}$$

$$\beta_{K^+K^+}^{f'} = -\sqrt{\frac{1}{20}}\beta_{\mathbf{8}T}\sin\theta_T + \sqrt{\frac{1}{8}}\beta_{\mathbf{1}T}\cos\theta_T = -\frac{1}{\sqrt{2}}\sqrt{\frac{3}{5}}\beta_{\mathbf{8}T} \tag{7c}$$

3. In this section, I only wrote the relative sign, not the relative magnitude given by $SU(2)$ and $SU(3)$ Clebsch-Gordan coefficients. In the $\pi^+\pi^+ \to \pi^+\pi^+$ case we obtain

$$0 = \beta^{f_+}(t) s^{\alpha_{f_+}(t)} - \beta^{\rho_-}(t) s^{\alpha_{\rho_-}(t)}. \tag{8}$$

Since this relation is valid in a range of s and t, we obtain $\alpha_{\rho_-}(t) = \alpha_{f_+}(t)$ and $\beta^{\rho_-}(t) = \beta^{f_+}(t)$. For particles with spin, one can repeat the argument with specific combination of helicity amplitudes having good naturality. Hence we obtain EXD relations between exchanges with the same naturality. In the case of $\pi^+\rho^+ \to \rho^+\pi^+$ case we obtain for the natural exchanges

$$0 = \beta^{\omega_-}(t)s^{\alpha_{\omega_-}(t)} - \beta^{a_+}(t)s^{\alpha_{a_+}(t)} + \beta^{h_+}(t)s^{\alpha_{h_+}(t)} - \beta^{\pi_-}(t)s^{\alpha_{\pi_-}(t)} \tag{9a}$$

$$= \left(\beta^{\omega_-}(t) - \beta^{a_+}(t)\right) s^{\alpha_N(t)} + \left(\beta^{h_+}(t) - \beta^{\pi_-}(t)\right) s^{\alpha_{EN}(t)}, \tag{9b}$$

and for the unnatural exchanges

$$0 = \beta^{\omega_+}(t)s^{\alpha_{\omega_+}(t)} - \beta^{a_-}(t)s^{\alpha_{a_-}(t)} + \beta^{h_-}(t)s^{\alpha_{h_-}(t)} - \beta^{\pi_+}(t)s^{\alpha_{\pi_+}(t)} \tag{9c}$$

$$= \left(\beta^{\omega_-}(t) - \beta^{a_+}(t) + \beta^{h_-}(t) - \beta^{\pi_+}(t)\right) s^{\alpha_U(t)}, \tag{9d}$$

In the reaction $\pi^+\pi^+ \to \rho^+\rho^+$, the exchanges pick up a sign equal to PC, we obtain

$$0 = \left(\beta^{\omega_-}(t) - \beta^{a_+}(t)\right) s^{\alpha_N(t)} + \left(\beta^{h_+}(t) - \beta^{\pi_-}(t)\right) s^{\alpha_{EN}(t)} \tag{9e}$$

$$0 = \left(\beta^{\omega_-}(t) - \beta^{a_+}(t) - \beta^{h_-}(t) + \beta^{\pi_+}(t)\right) s^{\alpha_U(t)} \tag{9f}$$

There are then EXD relation between exchanges with same naturality, same PC, same $G-$parity and opposite signature. The Regge trajectories are indicated on Fig. 1. The exchange degeneracy relations are summarized in Table 3 and in Fig. 1

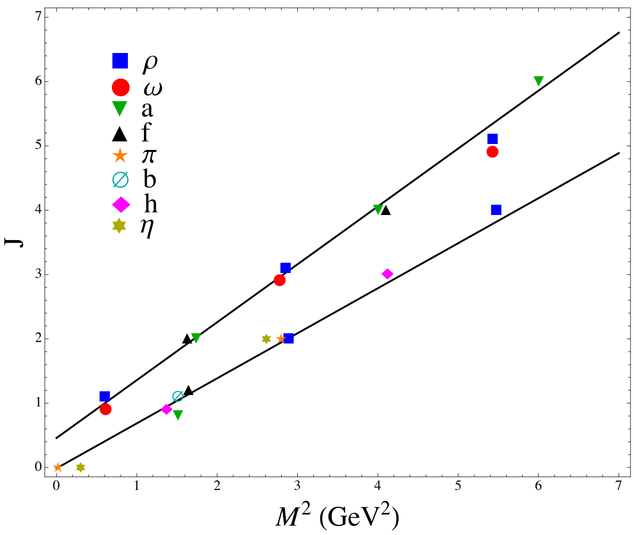

Figure 1: Regge trajectories.The solid lines are $\alpha_N(t) = 0.9(t - m_\rho^2) + 1$ and $\alpha_U(t) = 0.7(t - m_\pi^2) + 0$.

Table 3: Exchange degneracy relation

$\pi^+\pi^+ \to \pi^+\pi^+$	$\alpha_{f_+} = \alpha_{\rho_-}$	$\beta^{f_+}_{\pi^+\pi^+} = \beta^{\rho_-}_{\pi^+\pi^+}$
$K^+K^0 \to K^0K^+$	$\alpha_{a_+} = \alpha_{\rho_-}$	$\beta^{a_+}_{K^+K^0} = \beta^{\rho_-}_{K^+K^0}$
$K^+K^+ \to K^+K^+$	$\alpha_{f_+} = \alpha_{\omega_-}$	$\beta^{f_+}_{K^+K^+} = \beta^{\omega_-}_{K^+K^+}$
$K^+n \to K^+n$	$\alpha_{a_+} = \alpha_{\rho_-}$	$\beta^{a_+}_{pp} = \beta^{\rho_-}_{pp}$
$K^+p \to K^+p$	$\alpha_{f_+} = \alpha_{\omega_-}$	$\beta^{f_+}_{pp} = \beta^{\omega_-}_{pp}$
$\pi^+\rho^+ \to \rho^+\pi^+$	$\alpha_{h_+} = \alpha_{\pi_-}$	$\beta^{h_+}_{\pi^+\rho^+} = \beta^{\pi_-}_{\pi^+\rho^+}$
$\pi^+\pi^+ \to \rho^+\rho^+$	$\alpha_{h_-} = \alpha_{\pi_+}$	$\beta^{h_-}_{\pi^+\rho^+} = \beta^{\pi_+}_{\pi^+\rho^+}$
	$\alpha_{a_+} = \alpha_{\omega_-}$	$\beta^{a_+}_{\pi^+\rho^+} = \beta^{\omega_-}_{\pi^+\rho^+}$
	$\alpha_{a_-} = \alpha_{\omega_+}$	$\beta^{a_-}_{\pi^+\rho^+} = \beta^{\omega_+}_{\pi^+\rho^+}$

4. (a) For pion-nucleon scattering, the covariant basis is [3]

$$\langle N_j(p_4)\pi^b(p_3)|N_i(p_2)\pi^a(p_1)\rangle = \bar{u}(p_4)\left[A^{ba}_{ji} + (\not{p}_1 + \not{p}_3)B^{ba}_{ji}\right]u(p_2) \tag{10a}$$

$$A^{ba}_{ji} = \delta^{ba}\delta_{ji}A^{(+)} + i\epsilon^{abc}(\tau^c)_{ji}A^{(-)} \tag{10b}$$

$$A^{(\pm)}(\nu,t) = \pm A^{(\pm)}(-\nu,t) \tag{10c}$$

$$B^{(\pm)}(\nu,t) = \mp B^{(\pm)}(-\nu,t) \tag{10d}$$

The crossing variable is $\nu = (s-u)/2$ with $s = (p_1+p_2)^2$ and $u = (p_1-p_4)^2$. To derive the crossing relation, use C invariance $T = C^{-1}TC$, $v = -C\bar{u}^T$, $\bar{v} = \bar{u}^T C$, $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ and $C^{-1}\gamma_5 C = +\gamma_5^T$ and take the transpose complexe conjugate.
For kaon-nucleon scattering, the covariant basis is

$$\langle N_j(p_4)K^l(p_3)|N_i(p_2)K^k(p_1)\rangle = \bar{u}(p_4)\left[A^{lk}_{ji} + (\not{p}_1 + \not{p}_3)B^{lk}_{ji}\right]u(p_2) \tag{11}$$

and the isospin decomposition is

$$A^{lk}_{ji} = \delta^{lk}\delta_{ji}A^{(0)} + (\tau^a)_{ji}(\tau^a)_{kl}A^{(1)} \tag{12}$$

$A^{(0)}$ and $A^{(1)}$ have isospin 0 and 1 in the $t-$channel. The crossing relations are

$$A^{(0)}(-\nu,t) = +A^{(0)}(\nu,t) \qquad B^{(0)}(-\nu,t) = -B^{(0)}(\nu,t) \tag{13a}$$

$$A^{(1)}(-\nu,t) = -A^{(1)}(\nu,t) \qquad B^{(1)}(-\nu,t) = +B^{(1)}(\nu,t) \tag{13b}$$

(b) In nucleon-nucleon scattering there are five independent Lorentz structures. One possible solution is to use a $t-$channel base

$$\langle N_j(p_4)N_l(p_3)|N_i(p_2)N_k(p_1)\rangle = \sum_{n=1}^{5}(A_n)^{lk}_{ji}\ \bar{u}_3\phi^A_n u_1\ \bar{u}_2\phi^A_n u_4 \tag{14}$$

The index A is a collective representation of Lorentz indices. The tensor structures are

$$\phi_1 = 1 \qquad \phi_2 = \gamma_5 \qquad \phi_3^\mu = \gamma^\mu \qquad \phi_4^\mu = \gamma_5\gamma^\mu \qquad \phi_5^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] \tag{15a}$$

In this base, the scalar amplitudes A_n have good $t-$channel quantum numbers.
One could also use a $s-$channel base

$$\langle N_j(p_4)N_l(p_3)|N_i(p_2)N_k(p_1)\rangle = \sum_{n=1}^{5}(B_n)^{lk}_{ji}\ \bar{u}_2\phi^A_n u_1\ \bar{u}_3\phi^A_n u_4 \tag{16}$$

Fiertz identities relate the two basis. The transformation is

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 & -1/4 & -1/4 & 1/4 \\ 1 & -1/2 & 0 & -1/2 & -1 \\ -3/2 & 0 & -1/2 & 0 & -3/2 \\ -1 & -1/2 & 0 & -1/2 & 1 \\ 1/4 & -1/4 & -1/4 & 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} \tag{17}$$

The isospin decomposition is the same as in KN scattering and the crossing relations are

$$A_1^{(0,1,2)}(-\nu,t) = +A_1^{(0,1,2)}(-\nu,t) \tag{18a}$$

$$A_2^{(0,1,2)}(-\nu,t) = +A_2^{(0,1,2)}(-\nu,t) \tag{18b}$$

$$A_3^{(0,1,2)}(-\nu,t) = -A_3^{(0,1,2)}(-\nu,t) \tag{18c}$$

$$A_4^{(0,1,2)}(-\nu,t) = -A_4^{(0,1,2)}(-\nu,t) \tag{18d}$$

$$A_5^{(0,1,2)}(-\nu,t) = +A_5^{(0,1,2)}(-\nu,t) \tag{18e}$$

A_3 and A_4 pick up a minus sign because they correspond to negative signature exchanges (vector and axial-vector exchange). That's a good cross-check of the method.

(c) The reactions involve a vector with momentum p_V and polarization tensor $\epsilon_\mu(p_V,\lambda)$ and three pseudoscalar with momenta $p_{1,2,3}$. We need a Levi-Civita tensor ϵ for parity (an odd number of unnatural parity mesons) if parity is conserved. If parity is not conserved (weak decay) there are two additional structures. In the parity conserving decay $\omega \to \pi\pi\pi$ the Lorentz structure is

$$\langle \pi^a(p_1)\pi^b(p_2)\pi^a(p_3)|\omega(p_V,\lambda)\rangle = A^{abc}(\nu,t) i\varepsilon_{\alpha\beta\mu\nu}\epsilon^\alpha(p_V,\lambda)p_1^\beta p_2^\mu p_2^\nu. \tag{19}$$

The only isospin structure is $A^{abc}(\nu,t) = \varepsilon^{abc}A(\nu,t)$. Two pions are always in isospin 1. The scalar function is odd under crossing ($p_{1,2}\to p_{2,1}$ if $t=(p_V-p_3)^2$), $A(-\nu,t)=-A(\nu,t)$, since only vector are allowed.
The decay $B \to J/\psi K\pi$ can violate parity. There are then three Lorentz structures

$$\begin{aligned}\langle \pi(p_1)K(p_2)J/\psi(p_V)|B(p_1,\lambda)\rangle &= A_1(\nu,t) i\varepsilon_{\alpha\beta\mu\nu}\epsilon^\alpha(p_V,\lambda)p_1^\beta p_2^\mu p_2^\nu \\ &+ A_2(\nu,t)\epsilon^\mu(p_V,\lambda)(p_1-p_2)_\mu \\ &+ A_3(\nu,t)\epsilon^\mu(p_V,\lambda)(p_1+p_2)_\mu \end{aligned}\tag{20}$$

Isospin is not conserved, so the isospin structure is irrelevant. This base is relavant to study crossing under $p_{1,2}\to p_{2,1}$. We obtain $A_1(-\nu,t) = -A_1(\nu,t)$, $A_2(-\nu,t)=+A_2(\nu,t)$ and $A_3(-\nu,t) = -A_3(\nu,t)$. So the exchanges (or resonances) in the 12 channel are $(\eta=+,\tau=-)$ in A_1, $(\eta=+,\tau=+)$ in A_2 and $(\eta=-,\tau=-)$ in A_3.

(d) There are four independent structures. With the notation $P=(p_1+p_2)/2$, $\epsilon_1\equiv\epsilon(k_1,\lambda_1)$ and $\epsilon_2\equiv\epsilon(k_2,\lambda_2)$, they are

$$\begin{aligned}\langle \pi^d(p_2)\rho^c(k_2,\lambda_2)|\pi^a(p_1)\rho^b(k_1,\lambda_1)\rangle &= A_1^{abcd}(\nu,t)\ \epsilon_1\cdot\epsilon_2 \\ &+ A_2^{abcd}(\nu,t)\ P\cdot\epsilon_1\ P\cdot\epsilon_2 \\ &+ A_3^{abcd}(\nu,t)\left[k_2\cdot\epsilon_1\ P\cdot\epsilon_2 + P\cdot\epsilon_1\ k_1\cdot\epsilon_2\right] \\ &+ A_4^{abcd}(\nu,t)\ k_2\cdot\epsilon_1\ k_1\cdot\epsilon_2. \end{aligned}\tag{21}$$

The isospin decomposition is

$$A^{abcd} = \delta^{ac}\delta_{bd}A^{(0)} + \frac{1}{2}\left(\delta^{ab}\delta^{cd}-\delta^{ad}\delta^{bc}\right)A^{(1)} + \frac{1}{2}\left(\delta^{ab}\delta^{cd}+\delta^{ad}\delta^{bc}\right)A^{(2)} \tag{22}$$

The Lorentz and isospin bases are chosen to have good properties under crossing the two pions (or the two ρ's). We obtain, for $i=1,2,3,4$

$$A_i^{(0,2)}(-\nu,t) = +A_i^{(0,2)}(\nu,t) \qquad\qquad A_i^{(1)}(-\nu,t) = -A_i^{(1)}(\nu,t) \tag{23}$$

(e) The momenta are $\gamma^{(*)}(k)+N(p_1)\to\pi(q)+N(p_2)$ and $p=(p_1+p_2)/2$. Use $F_{\mu\nu}=\epsilon_\mu k_\nu - k_\mu\epsilon_\nu$. Parity requires a γ_5 or an $\varepsilon_{\alpha\beta\mu\nu}$. The matrix element is

$$\langle N_j(p_2)\pi^a(q)|\gamma(k)N_i(p_1)\rangle = \sum_n (A_n)^a_{ji} M_n \tag{24}$$

We found in the notation of Ref. [4]

$$M_1 = \frac{1}{2}\gamma_5\gamma_\mu\gamma_\nu F^{\mu\nu} \tag{25a}$$
$$M_2 = 2\gamma_5 q_\mu p_\nu F^{\mu\nu} \tag{25b}$$
$$M_3 = \gamma_5\gamma_\mu q_\nu F^{\mu\nu} \tag{25c}$$
$$M_4 = \frac{i}{2}\epsilon_{\alpha\beta\mu\nu}\gamma^\alpha q^\beta F^{\mu\nu} \tag{25d}$$
$$M_5 = \gamma_5\gamma^\mu k^\nu F_{\mu\nu} \tag{25e}$$
$$M_6 = \gamma_5 q^\mu k^\nu F_{\mu\nu} \tag{25f}$$

The last $M_{5,6}$ are zero for photoproduction. The isospin decomposition is

$$(A_n)^a_{ji} = A^{(+)}\delta^{a3}\delta_{ji} + A^{(-)}\frac{1}{2}[\tau^a, \tau^3]_{ji} + A^{(0)}\tau^a_{ji} \tag{26}$$

Finally the crossing properties are

$$A_i^{(0,+)}(-\nu,t) = +A_i^{(0,+)}(\nu,t) \qquad A_i^{(-)}(-\nu,t) = -A_i^{(-)}(\nu,t) \qquad i=1,2,4 \tag{27a}$$
$$A_3^{(0,+)}(-\nu,t) = -A_3^{(0,+)}(\nu,t) \qquad A_3^{(-)}(-\nu,t) = +A_3^{(-)}(\nu,t) \tag{27b}$$

References

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