Hadron-nucleus interactions at very high energies: black disk limit

Vadim Guzey

Petersburg Nuclear Physics Institute (PNPI),
National Research Center “Kurchatov Institute”,
Gatchina, Russia

Outline:

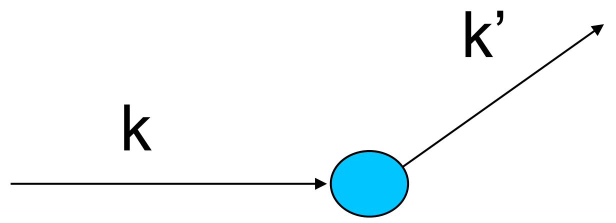
- Black disk limit in non-relativistic quantum mechanics
- Gribov derivation of black disk limit in hadron-nucleus scattering, V.N. Gribov, Sov. Phys. JETP 30 (1970) 709; Bertocchi, Nuovo Cim. A11 (1972) 45
- Black disk limit in deep inelastic scattering (DIS) on nuclei, V.N. Gribov, Sov. Phys. JETP 30 (1970) 709; Frankfurt, Guzey, McDermott, Strikman, Phys. Rev. Lett. 87 (2001) 192301

2017 International Summer Workshop on Reaction Theory
Bloomington, Indiana, USA, June 12-22, 2017

Black disk limit in quantum mechanics

- Black disk limit = high-energy scattering on a completely absorbing target
- Standard problem of non-relativistic (NR) quantum mechanics, Landau and Lifshitz, v.3
- Asymptotic form of wave function:

$$\Psi_k(\vec{r}) = e^{i\vec{k}\vec{r}} + f(\vec{k}', \vec{k})\,\frac{e^{ikr}}{r}$$



- Formal general solution of Schrodinger equation for scattering amplitude for potential V:

$$f(\vec{k}', \vec{k}) = -\frac{m}{2\pi\hbar^2}\int d^3\vec{r'}\, e^{-i\vec{k'}\vec{r'}}\, V(\vec{r'})\, \Psi_k(\vec{r'})$$

- Not very helpful since we still do not know the wave function $\Psi_k(r')$.

Black disk limit in quantum mechanics

- For scattering of fast particles, look for solution in form of plane wave modified by a slow-changing function ϕ(r):

$$\Psi_k(\vec{r}) = e^{i\vec{k}\vec{r}}\phi(\vec{r})$$

- Solving Schrodinger equation for ϕ(r) and taking large r:

$$\phi(\vec{r}) = e^{-\frac{i}{\hbar v}\int_{-\infty}^{z} dz' V(x,y,z')}$$

→ the phase that a fast particle with velocity v along e_z accumulates while crossing potential V

- Solution for scattering amplitude of a fast particle on potential V:

$$f(\vec{k}',\vec{k}) = -\frac{m}{2\pi\hbar^2}\int d^3\vec{r}\, e^{i(\vec{k}-\vec{k}')\vec{r}}\, V(\vec{r})\, e^{-\frac{i}{\hbar v}\int_{-\infty}^{z} dz' V(x,y,z')}$$

Black disk limit in quantum mechanics

- For fast particles, momentum transfer q=k-k' is transverse to particle momentum k → natural to separate transverse (b) and longitudinal (z) coordinates:

$$\vec{r} = \vec{b} + z e_z$$

- The scattering amplitude becomes:

$$f(\vec{k}', \vec{k}) = -\frac{m}{2\pi\hbar^2} \int d^2\vec{b}\, e^{i\vec{q}\vec{b}} \int_{-\infty}^{\infty} dz\, V(\vec{b}, z)\, e^{-\frac{i}{\hbar v} \int_{-\infty}^{z} dz' V(\vec{b}, z')}$$

$$= \frac{ik}{2\pi} \int d^2\vec{b}\, e^{i\vec{q}\vec{b}} \left[1 - e^{-\frac{i}{\hbar v} \int_{-\infty}^{\infty} dz' V(\vec{b}, z')}\right]$$

- Introducing the so-called eikonal phase:

$$\chi(b) \equiv -\frac{1}{\hbar v} \int_{-\infty}^{\infty} dz' V(\vec{b}, z')$$

$$f(\vec{k}', \vec{k}) = \frac{ik}{2\pi} \int d^2\vec{b}\, e^{i\vec{q}\vec{b}} \left[1 - e^{i\chi(b)}\right]$$

Black disk limit in quantum mechanics

- Thus, fast particles travel along essentially straight trajectories (geometrical optics) and accumulate eikonal phase χ depending on impact parameter b.

- Total cross section using optical theorem:

$$\sigma_{\rm tot} = \frac{4\pi}{k} Im f(\vec{k},\vec{k}) = 2\Re e \int d^2\vec{b}\,\left[1 - e^{i\chi(b)}\right]$$

- For spherical potential of radius R and depth V_0:

$$\chi(b) = \begin{cases} \frac{2V_0}{\hbar v}\sqrt{R^2 - b^2}\,, & b < R \\ 0\,, & b > R \end{cases}$$

- For completely absorbing potential $V_0 \to \infty$:

$$\sigma_{\rm tot} = 2\pi R^2$$

Black disk limit in quantum mechanics

- Thus, $\sigma_{tot}=2\pi R^2$ in the black disk limit → twice the geometric cross section, where πR^2 comes from full absorption and πR^2 — from elastic scattering off the sharp edge of potential.

- This is Babinet's (complementarity) principle of geometrical optics.

Black disk limit in quantum mechanics

- In this representation, it is very intuitive to formulate unitarity condition for the profile function Γ(b):

$$\Gamma(b) = 1 - e^{i\chi(b)}$$

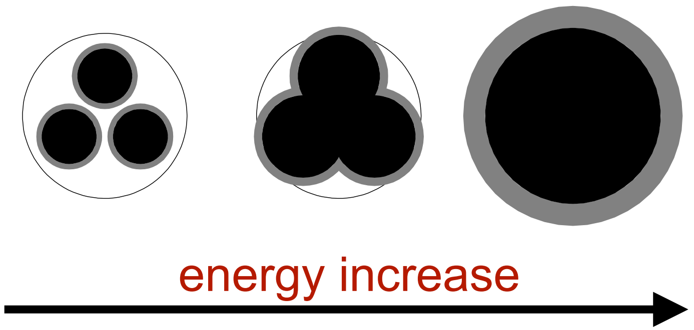
- The integrand of σ_{in} should not exceed unity → unitarity constrain on Γ(b) at given b (similarly to unitarity for partial waves f_l)

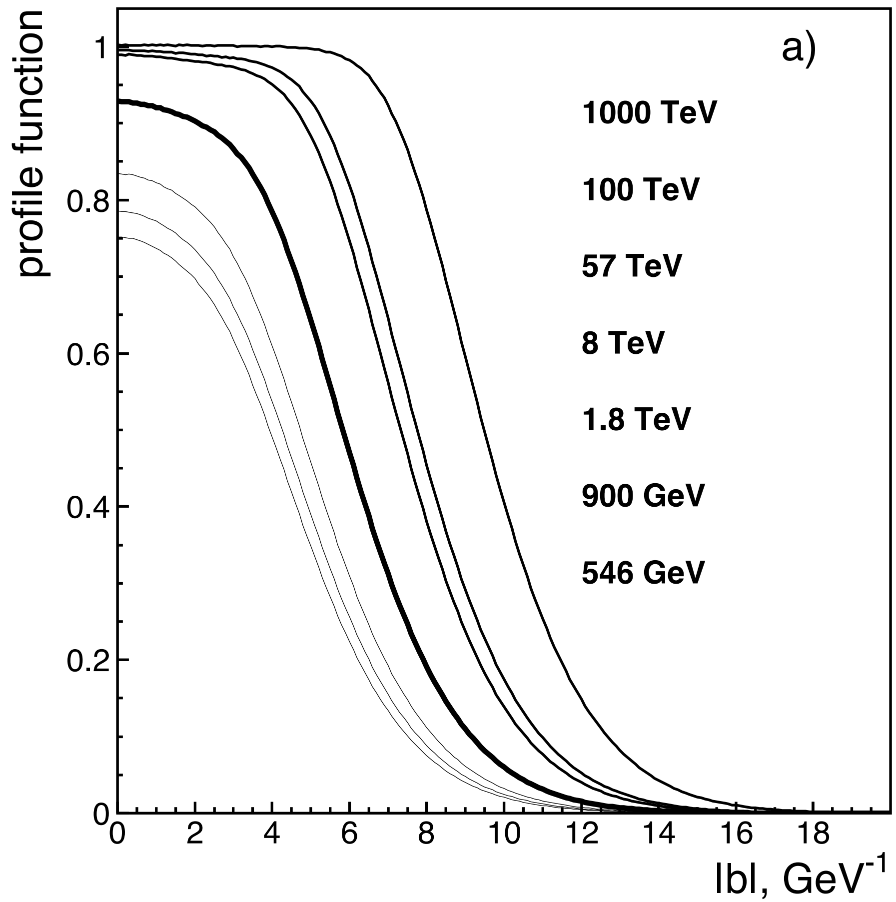
$$2\Re e\Gamma(b) - |\Gamma(b)|^2 \leq 1$$

- Powerful tool for analysis of very high-energy proton-proton and proton-nucleus scattering (Large Hadron Collider — LHC), and photon-proton and photon-nucleus scattering (HERA, Electron-Ion Collider, Large Hadron Electron Collider@CERN).

Black disk limit in quantum mechanics

• Γ(b) describes the proton/nucleus density (or profile) in transverse plane (impact parameter space) :





Anisovich et al, Int. J. Mod. Phys. A29 (2014) 1540096

Black disk limit in quantum mechanics

- In Gribov lectures "*Strong Interactions of Hadrons at High Energies*", black disk limit is derived using partial waves. The two methods are equivalent.

- Starting from standard partial wave decomposition of scattering amplitude,

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos\theta)$$

- using large-l asymptotic of Legendre polynomials,

$$P_l(\cos\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{-i\theta l \cos\phi}$$

- one can replace sum over l by integral over b=l/k, notice that q=kθ and show that

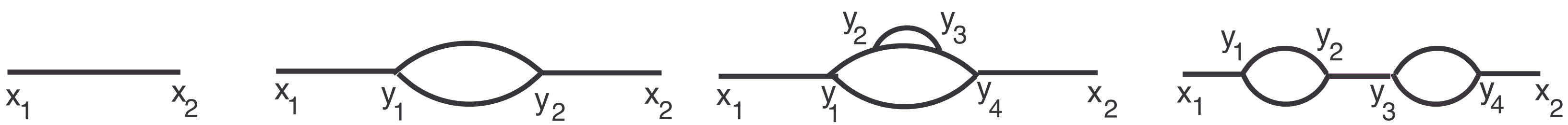
$$f(\theta) = \frac{k^2}{\pi} \int d^2b \, e^{-i\vec{q}\vec{b}} f_l$$

- The partial wave is then: $$f_l = \frac{i}{2k}\left[1 - e^{i\chi(b)}\right] = \frac{i}{2k}\Gamma(b)$$

Gribov space-time picture of hadron interactions at high energy

• From non-relativistic quantum mechanics → relativistic quantum field theory

• Hadrons are composite particles that can create virtual particles (fluctuations):



• Characteristic time of such fluctuation is 1/μ in rest frame (μ is pion mass proton beam) → large in the laboratory frame due to Lorentz time dilation:

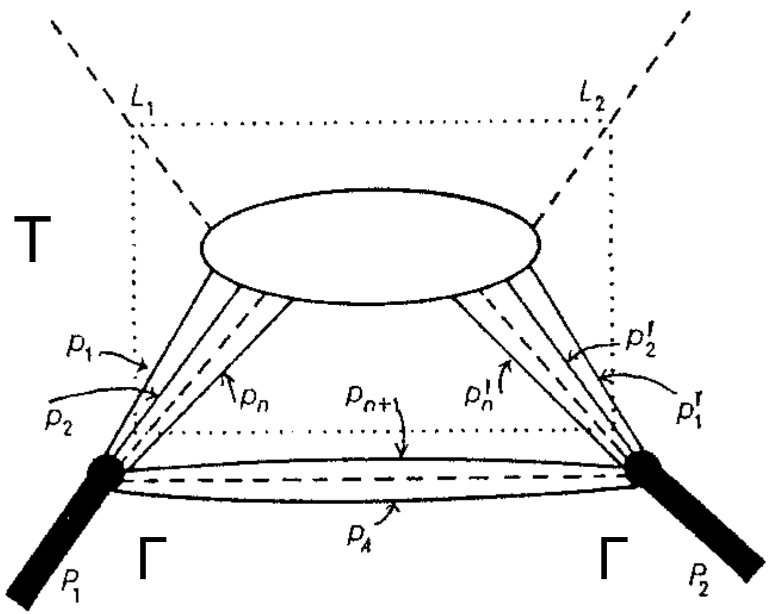
$$l_c \propto \frac{p}{\mu^2} \geq \text{target size}$$

• At high energies (E > 5-10 GeV), hadrons interact as superpositions of long-lived virtual particles coherently (simultaneously) with all nucleons of nuclear target, Gribov, Ioffe, Pomeranchuk, 1966; Gribov, arXiv: 0006158 → Gribov theory of nuclear shadowing, V.N. Gribov, Sov. Phys. JETP 29 (1969) 483 and black disk limit, V.N. Gribov, Sov. Phys. JETP 30 (1970) 709.

Hadron-nucleus scattering

- Consider interaction of high-energy hadron with n nucleons of a nuclear target containing A nucleons, V.N. Gribov, Sov. Phys. JETP 30 (1970) 709; Bertocchi, Nuovo Cim. A11 (1972) 45

- Corresponding effective Feynman graph: all coupling are strong → use to obtain the analytic structure of amplitudes



$$p_i = P_1/A + k_i$$
$$p'_j = P_2/A + k'_j$$
$$D(k) = (2\pi)^3(p^2 - m^2 + i\epsilon)$$

$$F^{(n)} = \frac{1}{n!(A-n)!}\prod_{i,j=1}^{A} d^4k_i\, d^4k'_j\, \delta^4(\sum_i^A p_i - P_1)\delta^4(\sum_i^A p'_j - P_2)$$

$$\times \prod_{m=n+1}^{A} \delta^4(P_1/A + k_m - P_2/A - k'_m)\frac{\Gamma(k_i)\Gamma(k'_j)T}{D(k_1)\ldots D(k_n)D(k'_1\ldots D(k'_n D(k_{n+1}\ldots D(k_A)}$$

Hadron-nucleus scattering

- Introduce relative and average momenta:

$$k_i = s_i + d_i/2$$
$$k'_i = s_i - d_i/2$$

energy-momentum conservation at nuclear vertices $\longrightarrow$

$$\sum s_i = 0$$
$$\sum d_i = 0$$

- For spectator nucleons:

$$p_m = p'_m = s_m$$
$$d_m = d/A = (P_1 - P_2)/A$$

- Assumption #1: non-relativistic nucleons in nuclear target.
- Assumption #2: all singularities are due to A+n-2 inverse propagators → integration over energies

$$F^{(n)} = \frac{1}{n!(A-n)!} \prod_{i \neq n}^{A} \frac{d^3 s_i}{2m(2\pi)^3} \prod_{j=1}^{n-1} \frac{d^3 d_i}{2m(2\pi)^3} \frac{\Gamma(s_i + d_i/2)\Gamma(s_i - d_i/2)T}{D(s_n + d_n/2)D(s_n - d_n/2)}$$

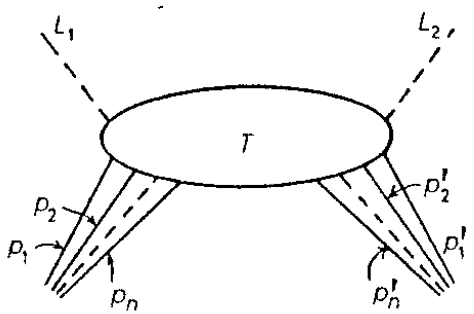
Hadron-nucleus scattering

- Coordinate space nucleus wave function:

$$\frac{\Gamma(p_i)}{D(s_n + d_n/2)} = \sqrt{A!(2m)^{A-1}} \prod_{i=1}^{A} d^3x_i \delta^3(\sum x_i) e^{-i\sum_i x_i p_i} \psi^*$$

$$\frac{\Gamma(p'_j)}{D(s_n - d_n/2)} = \sqrt{A!(2m)^{A-1}} \prod_{i=1}^{A} d^3y_i \delta^3(\sum y_i) e^{i\sum_i y_i p'_j} \psi$$

- Assumption #3: T depends weakly on s_i → corresponds to neglecting Fermi motion of nucleons or “frozen nucleon approximation”



$$F^{(n)} = \frac{A!}{n!(A-n)!} \frac{1}{(2m(2\pi)^3)^{n-1}} \prod_{i=1}^{A} \int d^3x_i |\psi|^2 \delta^3(\sum x_i) \prod_{i=1}^{n-1} d^3d_i T e^{-\sum_{i=1}^{A} x_i d_i}$$

Hadron-nucleus scattering

- Assumption #4: T does not depend of transverse component of momentum transfer d_i → nucleus size >> particle "size"
- Transverse and longitudinal momenta and coordinates:

$$d_i = t_i + q_i$$
$$x_i = b_i + z_i$$

- Trivial integration over t_i → all nucleons are at the same transverse position b → recall our non-relativistic case in the beginning.

$$F^{(n)} = \frac{A!}{(A-n)!}\frac{1}{(4\pi m)^{n-1}}\int d^2b\, e^{-ibd} e^{ibdA/n}\int_{-\infty}^{z_2} dz_1 \int_{-\infty}^{z_3} dz_2 \ldots \int_{-\infty}^{\infty} dz_n$$
$$\times\, \chi_n \prod_{i=1}^{n} dq_i \delta(\sum q_i) e^{-i\sum q_i z_i} T(q_i)$$

- Short-hand notation: $\chi(d,b,z_i) = \int d^3x_{n+1}\ldots x_A \delta(\sum_i x_i) e^{-id/A\sum_{k=n+1}^{A} x_k} |\psi|^2$

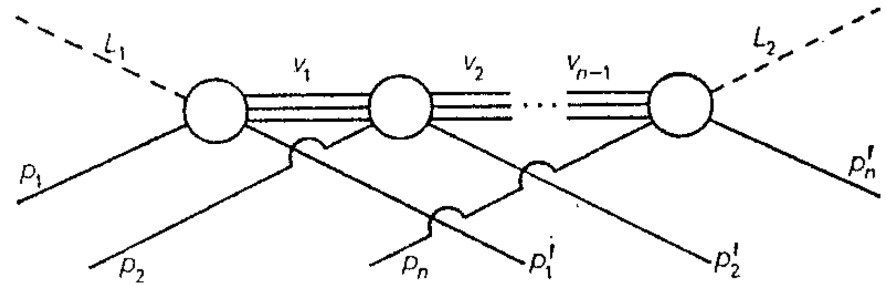
Hadron-nucleus scattering

- Note the space ordering $z_1 < z_2 \ldots < z_n \rightarrow$ need to pick up singularities of T
- To integrate over q_i, introduce momenta l_i:

$$l_1 = q_1 \, , \quad l_2 = q_1 + q_2 \, , \quad l_j = \sum_{i=1}^{j} q_i$$

$$\sum_i q_z z_i = l_1(z_1 - z_2) + l_2(z_2 - z_3) + \cdots + l_{n-1}(z_{n-1} - z_n)$$

- Multiparticle intermediate states for T, v_i is their four-momenta



- Longitudinal momenta l_i are linearly related to invariant mass of intermediate states:

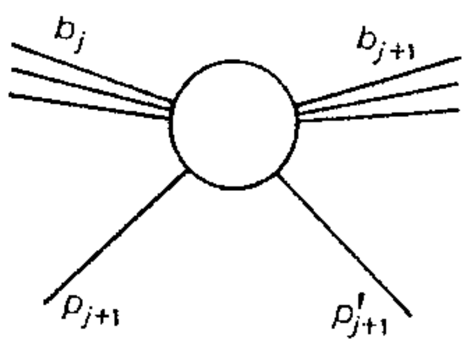
$$l_i = \frac{L_1^2 - v_i^2}{2p}$$

Hadron-nucleus scattering

• Assumption #5: singularities of T with respect to v_i^2 are isolated singularities (poles) → integration over l_i in upper half plane → corresponds to correct propagation of intermediate states:

$$F^{(n)} = \frac{A!}{(A-n)!}\frac{i}{(4mp)^{n-1}}\int d^2b e^{-ibd} e^{in/Abd}\int dz_1 \int dz_2 \ldots \int dz_n$$
$$\times \chi_n \sum_{b_1, b_2, \ldots b_{n-1}} f_{L_1 b_1} e^{-l_{b_1}(z_1 - z_2)} f_{b_1 b_2} e^{-il_{b_2}(z_2-z_3)} \ldots f_{b_{n-1} L_2}$$

• Interaction with n nucleons = sum over all possible intermediate state with mass $m_{b_i}^2 = m_{L_1}^2 - 2pl_{b_1}$ in terms of multiparticle-nucleon amplitudes



• Total amplitude: $F(p,d) = \sum_{n=1}^{A} F^{(n)}$

Hadron-nucleus scattering

- Within given approximations, we derived the general expression for hadron-nucleus scattering amplitude.
- To calculate the cross section off a heavy nucleus, one often takes A $\to\infty$ to obtain the optical model (approximation).
- The derivation is still non-trivial in the general case. As a simplifying example, we consider only *elastic intermediate states.*
- Assuming independent nucleons with density ρ(b,z):

$$F(p,d) = \sum_n F^{(n)} = A \int d^2b\, e^{-ibd} \int_{-\infty}^{\infty} dz_n \rho(b,z_n) f \exp\left[i \frac{Af}{4pm} \int_{-\infty}^{z_n} dz' \rho(b,z') \right]$$
$$= i4pm \int d^2b\, e^{-ibd} \left(1 - \exp\left[i \frac{Af}{4pm} \int_{-\infty}^{\infty} dz' \rho(b,z') \right] \right)$$

- Use the optical theorem: $\Im m f = 2mp\sigma_N \,, \quad \Im m F(p,d=0) = 2mp\sigma_A$

Hadron-nucleus scattering

- Final expression for the total hadron-nucleus cross section in optical model:

$$\sigma_A = 2\Re e \int d^2b \left(1 - e^{-A/2\sigma_N(1-i\eta)T_A(b)}\right)$$

nuclear optical density

$$T_A(b) = \int_{-\infty}^{\infty} dz \rho(b,z)$$

- Multiple interactions with target nucleons with imaginary amplitudes leads to destructive interference causing $\sigma_A < \sigma_N$, which is called nuclear shadowing.

- For complete absorption $\sigma_N \to \infty$:

$$\sigma_A = 2\pi R_A^2$$

Black disk limit

- Different regimes characterized by different A-dependence:

$\sigma_A \sim A$ → shadowing → BDL: $\sigma_A \sim A^{2/3}$

energy increase

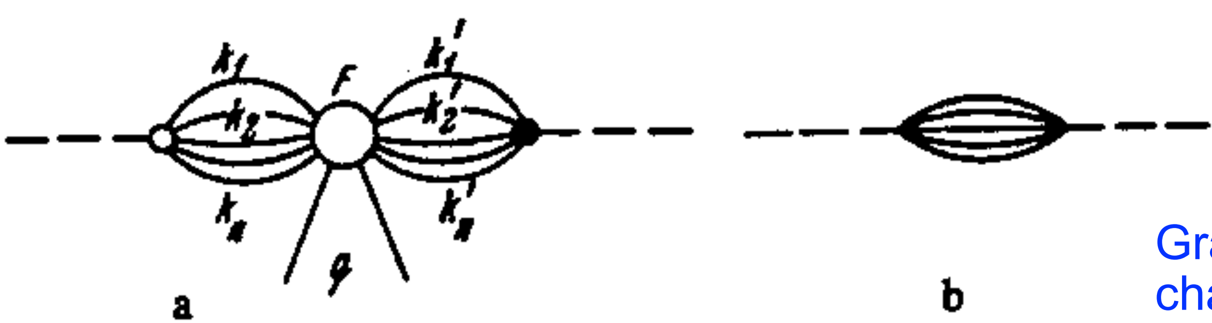
Photon-nucleus scattering

- Photons (real, virtual) take part in high-energy strong interactions through their hadronic component (fluctuations).
- Each of these fluctuations interacts with the BDL cross section $2\pi R_A^2$
- Total photon-nucleus cross section, Gribov, Sov. Phys. JETP 30 (1970) 709:

$$\sigma_{\gamma A} = 2\pi R_A^2 (1 - Z_3) = 2\pi R_A^2 e^2 \int \rho(M^2) \frac{dM^2}{M^2}$$

the fraction of time photon spends in its hadronic state in terms of spectral function

- Forward photon-nucleus amplitude= $i2\pi R_A^2 \prod \delta(k_i - k_i') = i2\pi R_A^2 \times (\text{graph b})$



Graph b determines the charge renormalization constant due to hadrons

Photon-nucleus scattering

• To formally derive this result for virtual photons, consider dispersion relation for forward Compton amplitude (recall the vector meson dominance model):

$$\sigma_{\gamma^* A} = \int\int \frac{dM^2 M^2}{Q^2 + M^2} \frac{dM'^2 M'^2}{Q^2 + M'^2} f_{\gamma^* V} \sigma_{VV'} f_{\gamma^* V'}$$

"vector meson" propagator

coupling constant

VM scattering cross section

• In BDL, off-diagonal transitions → 0 and $\sigma_V = 2\pi R_A^2$

• Also, note that $g^2_{\gamma^* V} \propto \sigma(e^+e^- \to \text{hadrons}) = \rho(M^2)/M^2$

$$\sigma_{\gamma^* A} = 2\pi R_A^2 e^2 \int \frac{dM^2 M^2}{(Q^2 + M^2)^2} \rho(M^2)$$

• This expression is logarithmically divergent at $M^2 \to 0$ due to infinite charge renormalization.

Photon-nucleus scattering

- To study its asymptotic behavior, recall that we always assume that:

$$l_c = \frac{\nu}{M^2} \geq R_A \quad \rightarrow \quad M^2 \leq \frac{\nu}{R_A} = \frac{Q^2}{2mxR_A}$$

notation for deep inelastic scattering: Q^2=photon virtuality; x=Bjorken x

- The virtual photon-nucleus cross section in black disk limit :

$$\sigma_{\gamma^* A} = 2\pi R_A^2 e^2 \rho(\infty) \ln(x_0/x)$$

- Dramatic violation of experimentally-observed approximate Bjorken scaling, $\sigma_{\gamma^* A} \sim 1/Q^2$, and much slower x-dependence.

- For the proton target, the x-dependence is faster due to diffractive cone shrinkage $\rightarrow$ $b_{max} \sim \ln(1/x)$,

$$\sigma_{\gamma^* p} = 2\pi R_N^2 e^2 \rho(\infty) \left(1 + c_N \ln^2(x_0/x)\right) \ln(x_0/x)$$

Photon-nucleus scattering

• In practice, the total cross section is not sensitive to BDL since it occurs only for a small fraction of all relevant fluctuations.

• BDL should be easier to see in diffractive processes, Frankfurt, Guzey, McDermott, Strikman, Phys. Rev. Lett. 87 (2001) 192301

• Using the guiding principle that in BDL, $\sigma_{diff}=1/2\sigma_{tot}$ and there are no off-diagonal transition, one can predict *diffractive structure functions*, which can be measured in the future in γ^*A DIS at Electron-Ion Collider:

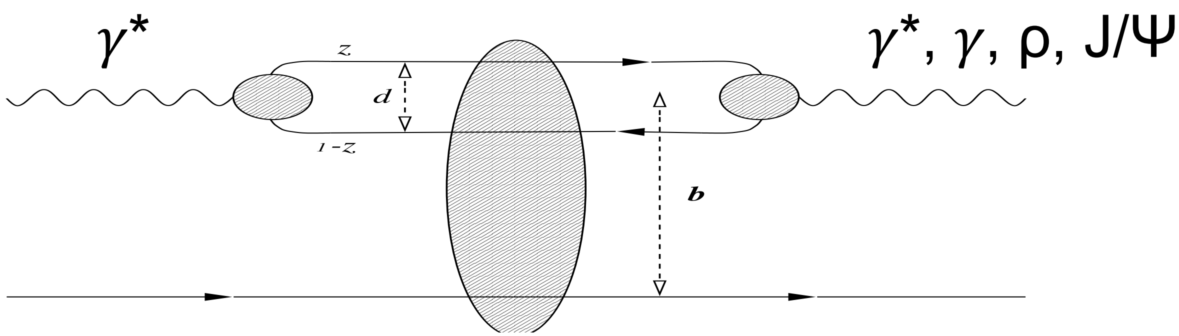
$$\frac{dF_T^{D(3)}(x,Q^2,M^2)}{dM^2} = \frac{\pi R_A^2}{12\pi^3}\frac{Q^2\rho(M^2)}{(M^2+Q^2)^2}$$

• One also predicts enhanced production of diffractive jets with large pT and “restoration” of VMD for electroproduction of vector mesons → both in stark contrast with usual leading-twist approximation:

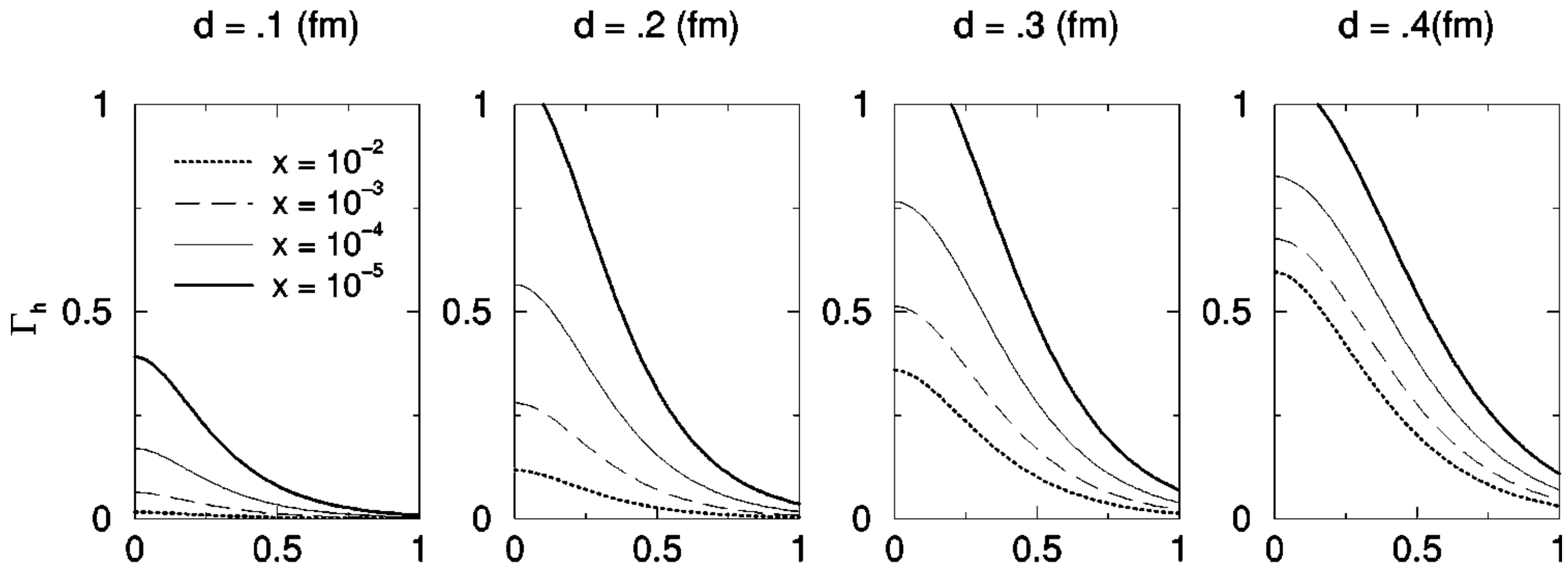
$$\frac{d\sigma^{\gamma_T^*+A\to V+A}}{dt} = \frac{M_V^2}{Q^2}\frac{d\sigma^{\gamma_L^*+A\to V+A}}{dt} = \frac{(2\pi R_A^2)^2}{16\pi}\frac{3\Gamma_V M_V^3}{\alpha(M_V^2+Q^2)^2}\frac{4|J_1(\sqrt{-t}\,R_A)|^2}{-tR_A^2}$$

BDL and color dipole phenomenology

- In phenomenological studies of BDL (saturation) in photon-hadron scattering at high energies, one often uses the color dipole model:



- Analysis of HERA data on t-dependence of $\gamma^* p \to J/\Psi p \to$ possibility to reconstruct the dipole profile function Γ(b) and study its proximity to BDL Γ ≈ 1, Rogers et al, Phys, Rev, D69 (2004) 074011:



Summary

• The black disk limit (BDL) of hadron (photon)-nucleus scattering is characterized by complete absorption for central (all) impact parameters leading to $\sigma_{tot}=2\pi R_T^2$.

• In pp scattering at the LHC, the proximity to BDL is examined using the profile function $\Gamma(b)$ in the impact parameter space.

• In DIS off nuclei, BDL signals violation of approximation Bjorken scaling of $\sigma_{\gamma^* A}$ and its slow, logarithmic x-dependence.

• It is suggested in the literature that a promising way to look for BDL (saturation) in DIS on nuclei is to study inclusive and exclusive diffraction.

• BDL is an important subject in view of ongoing efforts to strengthen the physics case for a future Electron-Ion Collider in the US.

• It is also relevant for studies of photon-nucleus scattering at high-energy in ultraperipheral collisions of ions at the LHC.