INDIANA UNIVERSITY
BLOOMINGTON

Jefferson Lab

Dispersion relations: some applications
Light quark masses from $\eta \to 3\pi$

Emilie Passemar

Indiana University/Jefferson Laboratory

epassema@indiana.edu

1.3 QCD at low energy

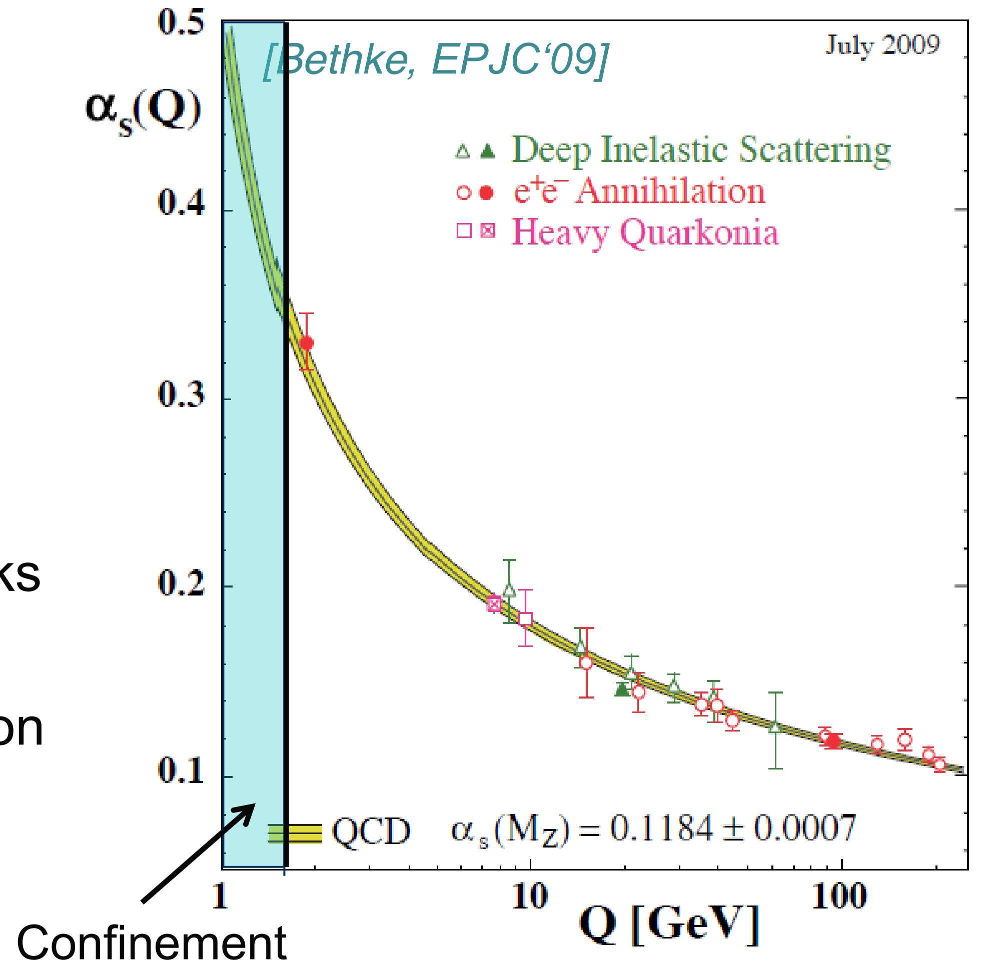
- At low energy, impossible to describe QCD with perturbation theory since α_S becomes large

  ➡ Need non perturbative methods

- Two model independent methods:
  - Effective field theory

    Ex: ChPT for light quarks

  - Numerical simulations on the lattice



1.4 Chiral Symmetry

- Limit $m_k \to 0$

$$\mathcal{L}_{QCD} \to \boxed{\mathcal{L}^0_{QCD} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \bar{q}_L i\gamma^\mu D_\mu q_L + \bar{q}_R i\gamma^\mu D_\mu q_R}\ , \quad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

with $q_{L/R} \equiv \frac{1}{2}(1 \mp \gamma_5) q$

Symmetry: $\boxed{G \equiv SU(3)_L \otimes SU(3)_R \to SU(3)_V}$

- G spontaneously broken, ground state not invariant under $G \equiv SU(3)_L \otimes SU(3)_R$ but invariant under $SU(3)_{V=L+R}$

⇨ Goldstone bosons with quantum numbers of pseudoscalar mesons are generated

Goldstone's Theorem

⇨ $\pi^+, \pi^0, \pi^-, K^+, K^0, \bar{K}^0, K^-, \eta$ massless states

1.5 Construction of an effective theory: ChPT

- Degrees of freedom: Goldstone bosons (GB)

  Symmetry group: $G \equiv SU(3)_L \otimes SU(3)_R$

- Build all the corresponding invariant operators including explicit symmetry breaking parameters

  ⇨ $\mathcal{L}_{ChPT} \equiv \mathcal{L}(U, \chi)$

  U: GB's χ: Masses ~ m_q

- Goldstone bosons interact weakly at low energy and $m_u, m_d \ll m_s < \Lambda_{QCD}$

  ⇨ expansion organized in external momenta and quark masses

Weinberg's power counting rule

$$\mathcal{L}_{eff} = \sum_{d \geq 2} \mathcal{L}_d \ , \mathcal{L}_d = \mathcal{O}\left(p^d\right) \ , p \equiv \left\{q, m_q\right\}$$

$$\mathbf{p} \ll \Lambda_H = 4\pi F_\pi \sim 1\ \mathrm{GeV}$$

1.6 Chiral expansion

- $$\mathcal{L}_{ChPT} = \underbrace{\mathcal{L}_2}_{\text{LO}:\ \mathcal{O}(p^2)} + \underbrace{\mathcal{L}_4}_{\text{NLO}:\ \mathcal{O}(p^4)} + \underbrace{\mathcal{L}_6}_{\text{NNLO}:\ \mathcal{O}(p^6)} + \ldots.$$

- Renormalizable and unitary order by order in the expansion

- The structure of the lagrangian is fixed by chiral symmetry but not the coupling constants → LECs appearing at each order

$$\mathcal{L}_2 :\ F_0, B_0, \qquad \mathcal{L}_4 = \sum_{i=1}^{10} L_i\, O_4^i, \qquad \mathcal{L}_6 = \sum_{i=1}^{90} C_i\, O_6^i$$

- LECs describe the influence of heavy degrees of freedom not contained in the ChPT lagrangian

- Naturalness: LECs of order one

1.6 Chiral expansion

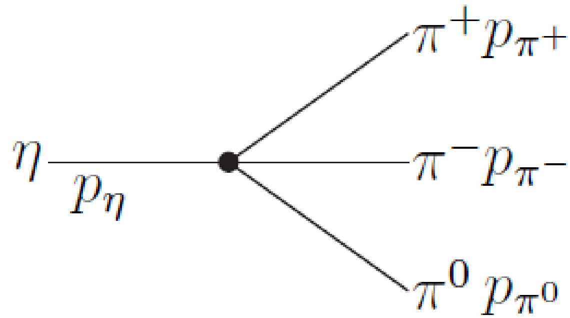
- The LECs calculable if QCD solvable, instead
  - Determined from experimental measurement
  - Estimated with models: Resonances, large N_C
  - Computed on the lattice

- In a specific process, only a limited number of LECs appear

1.6 Chiral expansion

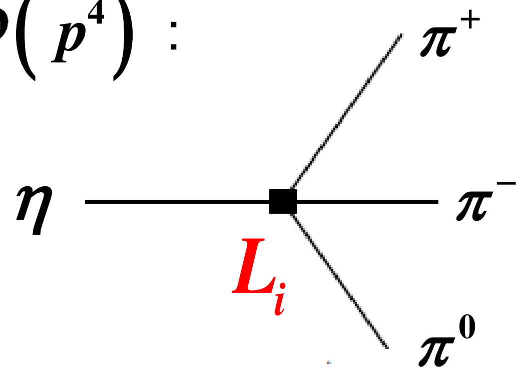
- Ex : η → $\pi^+ \pi^- \pi^0$ ⇨ $A = A_2 + A_4 + A_6 + \ldots$

  - Tree level $\mathcal{O}(p^2)$:



A_2

  - One loop $\mathcal{O}(p^4)$:



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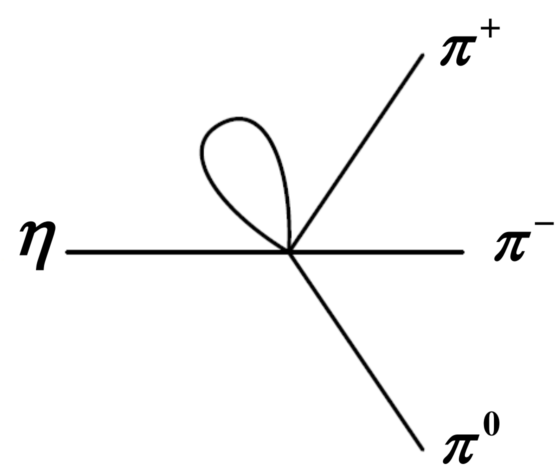

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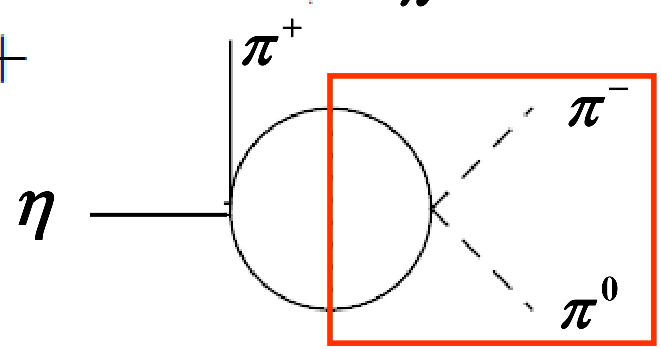


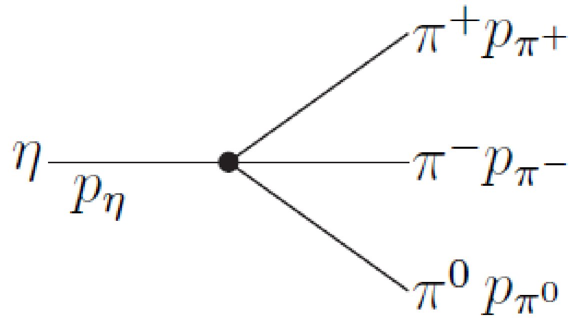
$\pi\pi \to \pi\pi$ at tree

A_4

1.6 Chiral expansion

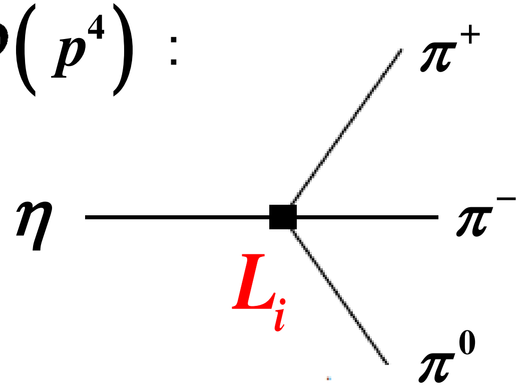
- Ex : $\eta \to \pi^+ \pi^- \pi^0$ ⇨ $A = A_2 + A_4 + A_6 + \ldots$

  - Tree level $\mathcal{O}(p^2)$:



A_2

  - One loop $\mathcal{O}(p^4)$:



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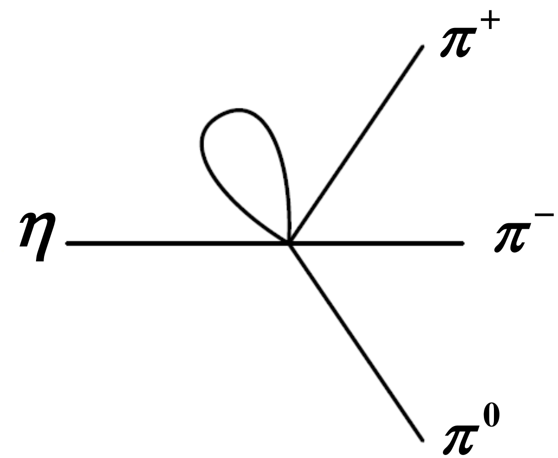

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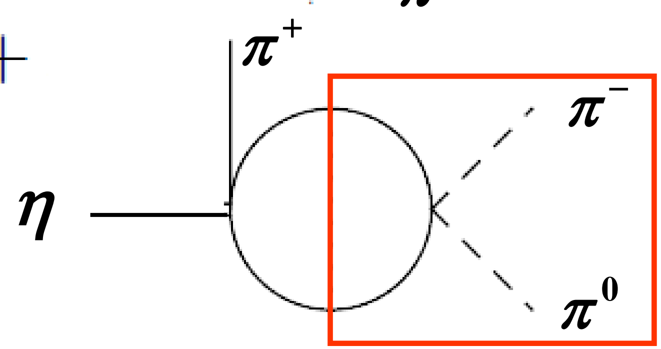
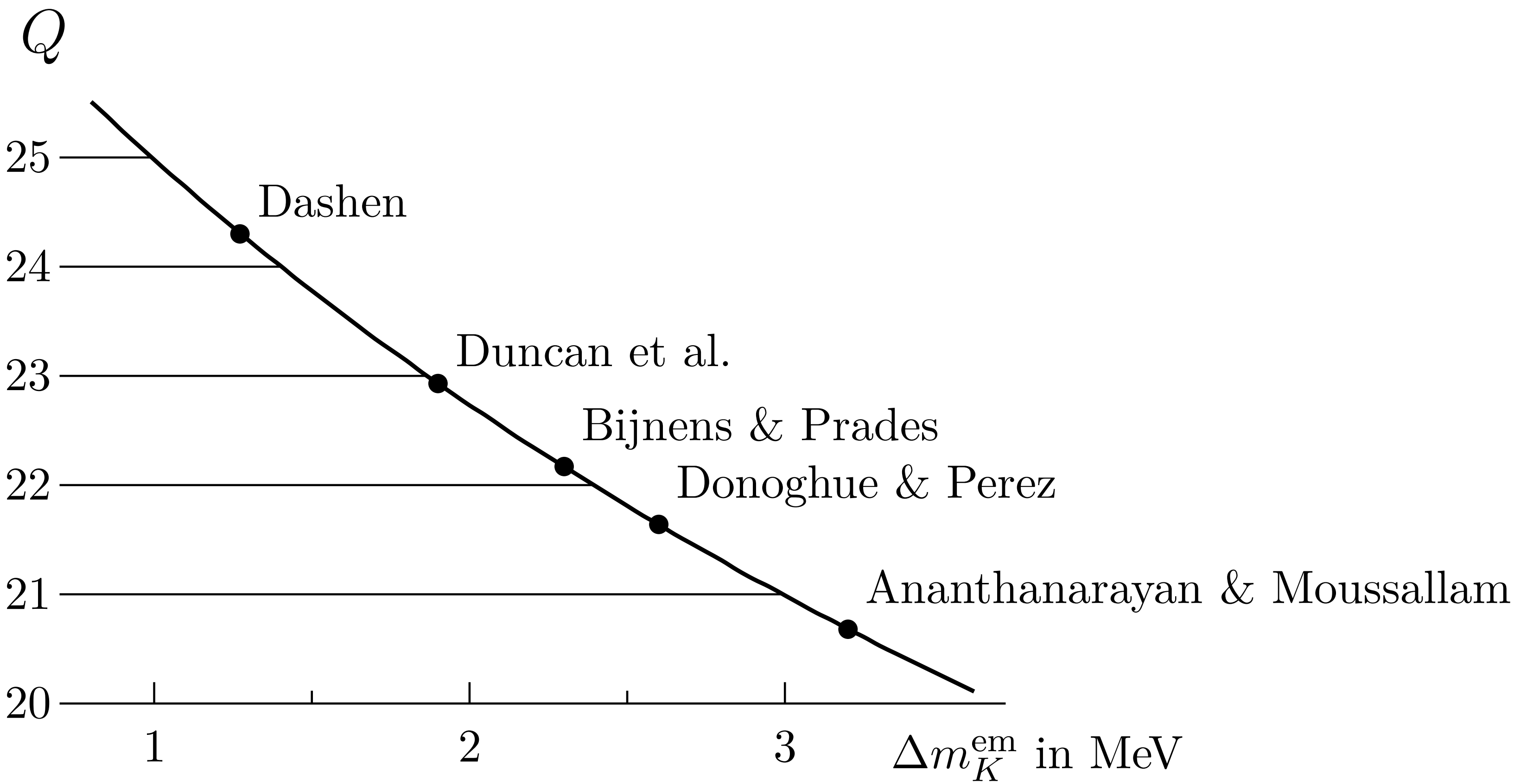


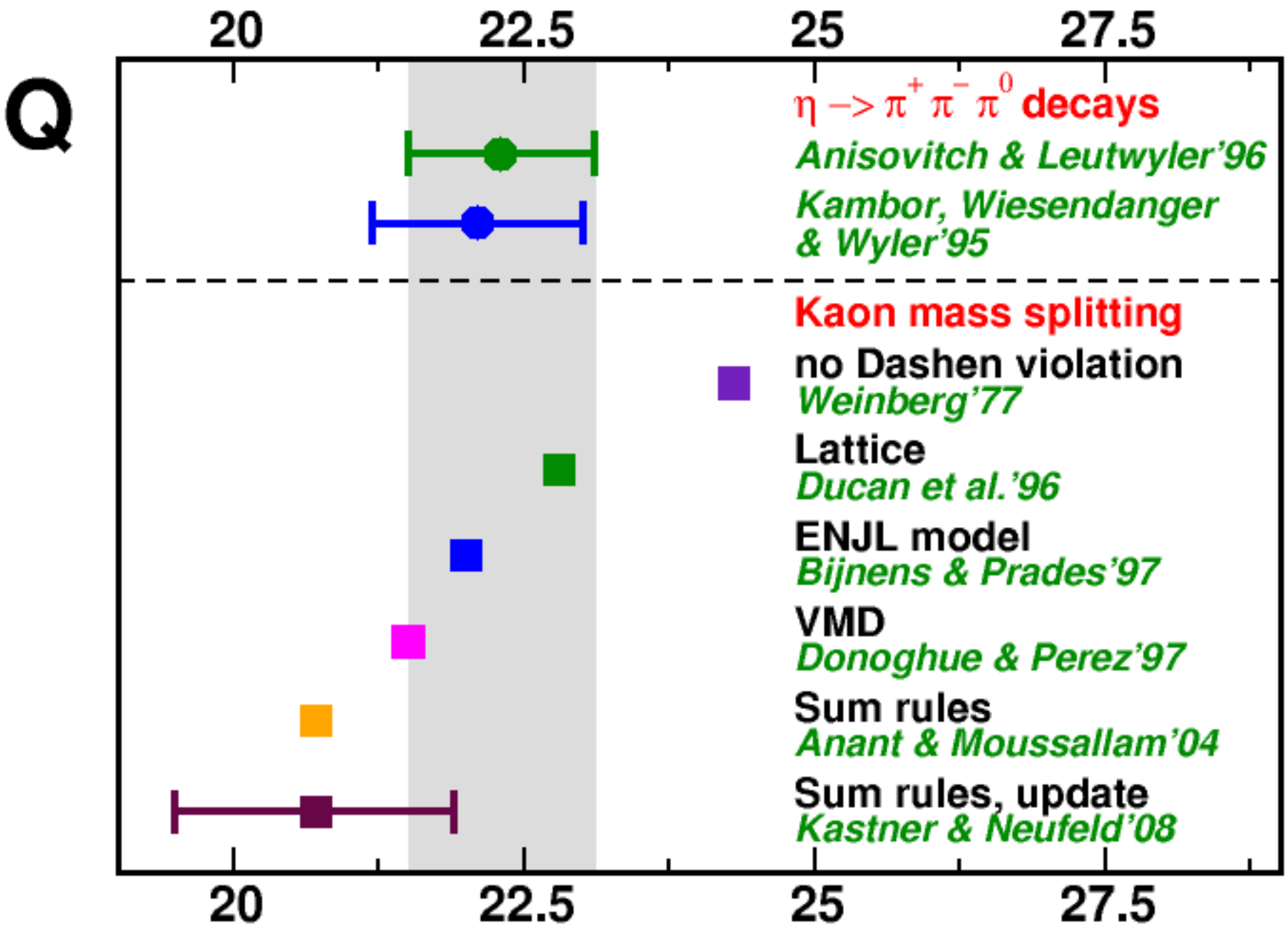
$\pi\pi \to \pi\pi$ at tree

A_4

Comparison of values of Q from Dashen corrections



Comparison of values of Q



$Q = 22.3 \pm 0.9$

$Q = 22.1 \pm 0.9$

$Q = 24.2$

$Q = 22.8$

$Q = 22$

$Q = 21.5$

$Q = 20.7$

$Q = 20.7 \pm 1.2$

➡ Fair agreement with the determination from meson masses

η

$I^G(J^{PC}) = 0^+(0^{-+})$

Mass $m = 547.862 \pm 0.018$ MeV

Full width $\Gamma = 1.31 \pm 0.05$ keV

C-nonconserving decay parameters

$\pi^+\pi^-\pi^0$ left-right asymmetry $= (0.09^{+0.11}_{-0.12}) \times 10^{-2}$

$\pi^+\pi^-\pi^0$ sextant asymmetry $= (0.12^{+0.10}_{-0.11}) \times 10^{-2}$

$\pi^+\pi^-\pi^0$ quadrant asymmetry $= (-0.09 \pm 0.09) \times 10^{-2}$

$\pi^+\pi^-\gamma$ left-right asymmetry $= (0.9 \pm 0.4) \times 10^{-2}$

$\pi^+\pi^-\gamma$ β (D-wave) $= -0.02 \pm 0.07$ (S = 1.3)

CP-nonconserving decay parameters

$\pi^+\pi^- e^+ e^-$ decay-plane asymmetry $A_\phi = (-0.6 \pm 3.1) \times 10^{-2}$

Dalitz plot parameter

$\pi^0\pi^0\pi^0$ $\alpha = -0.0315 \pm 0.0015$

η **DECAY MODES**		Fraction (Γ_i/Γ)	Scale factor/ Confidence level	p (MeV/c)
2γ		(39.41 ± 0.20) %	S=1.1	274
$3\pi^0$		(32.68 ± 0.23) %	S=1.1	179
$\pi^0 2\gamma$		$(2.7\ \pm0.5\ )\times10^{-4}$	S=1.1	257
$2\pi^0 2\gamma$		$<1.2\times10^{-3}$	CL=90%	238
4γ		$<2.8\times10^{-4}$	CL=90%	274
invisible		$<1.0\times10^{-4}$	CL=90%	–
Charged modes				
charged modes		(28.10 ± 0.34) %	S=1.2	–
$\pi^+\pi^-\pi^0$		(22.92 ± 0.28) %	S=1.2	174
$\pi^+\pi^-\gamma$		(4.22 ± 0.08) %	S=1.1	236
$e^+e^-\gamma$		$(6.9\ \pm0.4\ )\times10^{-3}$	S=1.3	274
$\mu^+\mu^-\gamma$		$(3.1\ \pm0.4\ )\times10^{-4}$		253
e^+e^-		$<5.6\times10^{-6}$	CL=90%	274
$\mu^+\mu^-$		$(5.8\ \pm0.8\ )\times10^{-6}$		253
$2e^+2e^-$		$(2.40\pm0.22)\times10^{-5}$		274
$\pi^+\pi^-e^+e^-(\gamma)$		$(2.68\pm0.11)\times10^{-4}$		235
$e^+e^-\mu^+\mu^-$		$<1.6\times10^{-4}$	CL=90%	253
$2\mu^+2\mu^-$		$<3.6\times10^{-4}$	CL=90%	161
$\mu^+\mu^-\pi^+\pi^-$		$<3.6\times10^{-4}$	CL=90%	113
$\pi^+e^-\overline{\nu}_e+$ c.c.		$<1.7\times10^{-4}$	CL=90%	256
$\pi^+\pi^-2\gamma$		$<2.1\times10^{-3}$		236
$\pi^+\pi^-\pi^0\gamma$		$<5\times10^{-4}$	CL=90%	174
$\pi^0\mu^+\mu^-\gamma$		$<3\times10^{-6}$	CL=90%	210
Charge conjugation (C), Parity (P), Charge conjugation × Parity (CP), or Lepton Family number (LF) violating modes				
$\pi^0\gamma$	C	$<9\times10^{-5}$	CL=90%	257
$\pi^+\pi^-$	P,CP	$<1.3\times10^{-5}$	CL=90%	236
$2\pi^0$	P,CP	$<3.5\times10^{-4}$	CL=90%	238
$2\pi^0\gamma$	C	$<5\times10^{-4}$	CL=90%	238
$3\pi^0\gamma$	C	$<6\times10^{-5}$	CL=90%	179
3γ	C	$<1.6\times10^{-5}$	CL=90%	274
$4\pi^0$	P,CP	$<6.9\times10^{-7}$	CL=90%	40
$\pi^0e^+e^-$	C [f]	$<4\times10^{-5}$	CL=90%	257
$\pi^0\mu^+\mu^-$	C [f]	$<5\times10^{-6}$	CL=90%	210
$\mu^+e^-+\mu^-e^+$	LF	$<6\times10^{-6}$	CL=90%	264

43. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

m_1	m_2	1, +1
+1/2	+1/2	1

m_1	m_2	1, 0	0, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1, −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2, +3/2
+1	+1/2	1

m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2, −3/2
−1	−1/2	1

2×1

m_1	m_2	3, +3
+2	+1	1

m_1	m_2	3, +2	2, +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3, −1	2, −1	1, −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3, −2	2, −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3, −3
−2	−1	1

1×1

m_1	m_2	2, +2
+1	+1	1

m_1	m_2	2, +1	1, +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2, −1	1, −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2, −2
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

3/2×1

m_1	m_2	5/2, +5/2
+3/2	+1	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2, −3/2	3/2, −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2, −5/2
−3/2	−1	1

2×1/2

m_1	m_2	5/2, +5/2
+2	+1/2	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2, −5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

4.2 Method: Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)$$

Fuchs, Sazdjian & Stern'93

Anisovich & Leutwyler'96

  - M_I isospin *I* rescattering in two particles
  - Amplitude in terms of S and P waves ⇨ exact up to NNLO ($\mathcal{O}(p^6)$)
  - Main two body rescattering corrections inside M_I

- Functions of only one variable with only right-hand cut of the partial wave ⇨ $disc\left[M_I(s)\right] \equiv disc\left[f_\ell^I(s)\right]$

- Elastic unitarity *Watson's theorem*

$$disc\left[f_\ell^I(s)\right] \propto t_\ell^*(s) f_\ell^I(s)$$

with $t_\ell(s)$ partial wave of elastic [illegible] scattering

4.4 Dispersion Relations for the $M_I(s)$

- Elastic Unitarity

$$[\mathrm{l}=1 \text{ for } I=1,\ \mathrm{l}=0 \text{ otherwise}]$$

$$\Rightarrow\quad disc\left[M_I\right]=disc\left[f_{\mathrm{l}}^I(s)\right]=\theta\left(s-4M_\pi^2\right)\left[M_I(s)+\hat{M}_I(s)\right]\sin\delta_{\mathrm{l}}^I(s)e^{-i\delta_{\mathrm{l}}^I(s)}$$

ππ phase shift

δ_{l}^I phase of the partial wave $f_{\mathrm{l}}^I(s)$

⇒ Watson theorem: elastic ππ scattering phase shifts

- Solution: Inhommogeneous Omnès problem

$$M_0(s)=\Omega_0(s)\left(\alpha_0+\beta_0 s+\gamma_0 s^2+\frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0^0(s')\,\hat{M}_0(s')}{\left|\Omega_0(s')\right|\left(s'-s-i\varepsilon\right)}\right)$$

Omnès function

$$\left[\Omega_I(s)=\exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{\delta_{\mathrm{l}}^I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

Similarly for M_1 and M_2

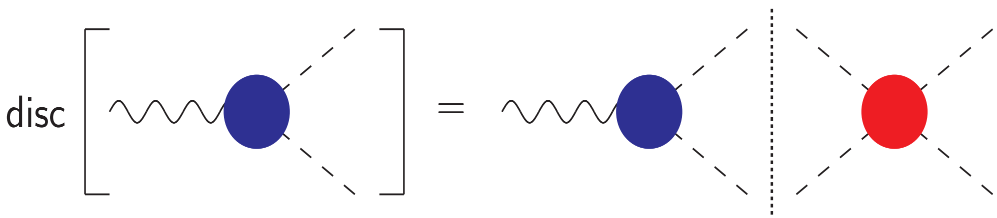
3.3 Dispersion Relations for the $M_I(s)$

- Unitary relation for $M_I(s)$:

$$disc\, M_I(s) = 2i\,\left(M_I(s) + \quad\right) t^*_{\pi\pi\to\pi\pi}(s)\,\rho(s)\;\theta\left(s-4M_\pi^2\right)$$

right-hand cut

Elastic Unitarity



$$\Rightarrow\quad t^*_{\pi\pi\to\pi\pi}(s)\,\rho(s) = \sin\delta_I(s)\,e^{-i\delta_I(s)}$$

- Right-hand cut only ⇒ Omnès problem

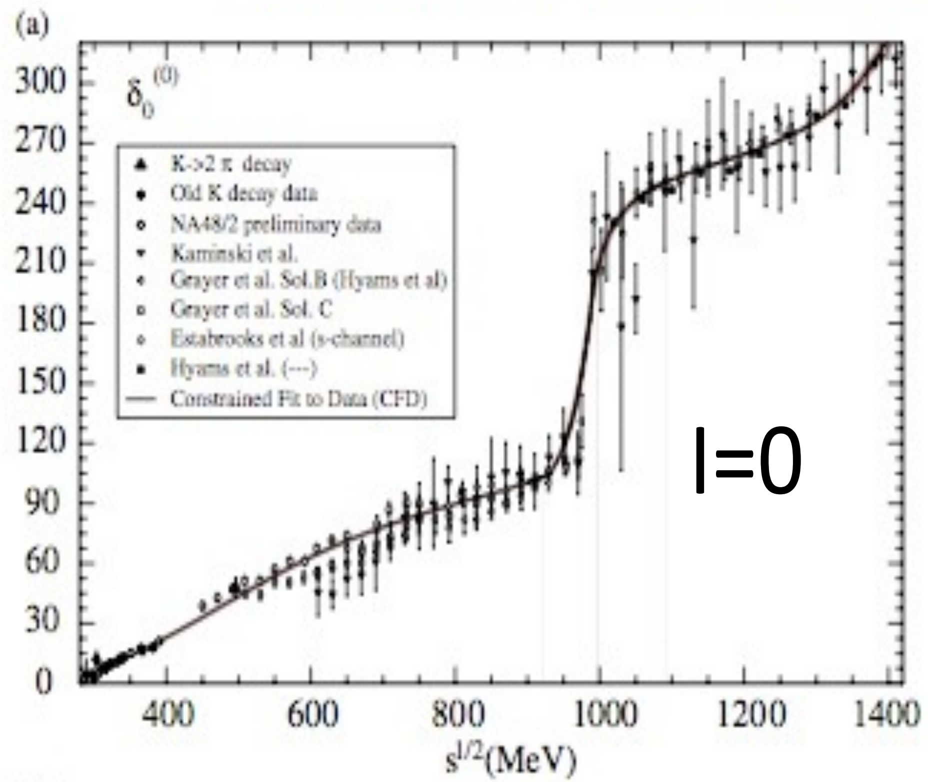
$$M_I(s) = P_I(s)\,\Omega_I(s) \qquad \left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

- *Watson's theorem* in the elastic region: Inputs needed : S and P-wave phase shifts of *$\pi\pi$ scattering*

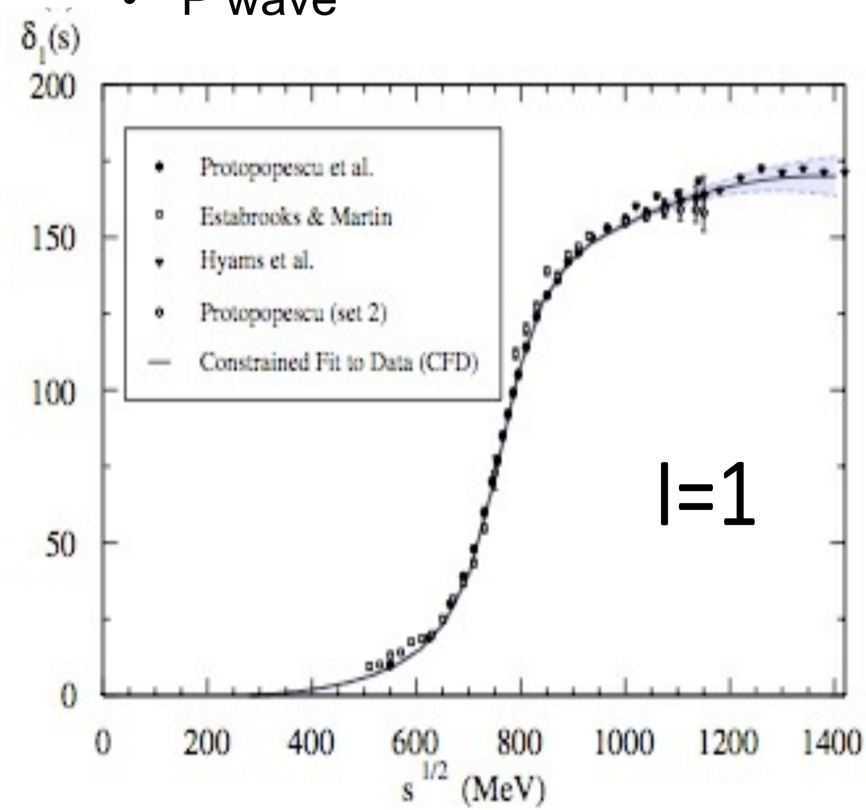
Inputs: $\pi\pi$ scattering

Garcia-Martin et al.'11

- S wave



- P wave



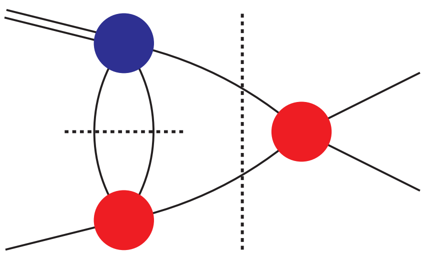
- $\pi\pi$ phase shifts extracted combining all experimental results solving Roy equations ⇨ A large number of theoretical analyses *Ananthanarayan et al'01, Colangelo et al'01, Descotes-Genon et al'01, Garcia-Martin et al'09,'11, Colangelo et al.'11* and all agree

3.3 Dispersion Relations for the $M_I(s)$

- Unitary relation for $M_I(s)$:

$$disc\ M_I(s) = 2i\left(M_I(s) + \hat{M}_I(s)\right)\ \sin\delta_I(s) e^{-i\delta_I(s)}\theta\left(s-4M_\pi^2\right)$$

right-hand cut — left-hand cut



- Dispersion relation for the M_I's

$$M_I(s) = \Omega_I(s)\left(P_I(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_I(s')\ \hat{M}_I(s')}{\left|\Omega_I(s')\right|\left(s'-s-i\varepsilon\right)}\right)$$

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

Omnès function

- $\hat{M}_I(s)$: singularities in the t and u channels, depend on the other $M_I(s)$
  Crossed-channel scattering between s-, t-, and u-channel
  Angular averages of the other functions
  ⇨ Coupled equations

Khuri & Treiman'60
Aitchison'77
Anisovich & Leutwyler'98

Hat functions

- Subtract $M_I(s)$ from the partial wave projection of $M(s,t,u)$

- Ex: $\hat{M}_0(s) = \frac{2}{3}\langle M_0 \rangle + 2(s-s_0)\langle M_1 \rangle + \frac{20}{9}\langle M_2 \rangle + \frac{2}{3}\kappa(s)\langle z M_1 \rangle$

where $\langle z^n M_I \rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z)), \quad z = \cos\theta$ scattering angle

Non trivial angular averages ⇨ need to deform the integration path to avoid crossing cuts
Generates complex analytic structure (3-particle cuts)

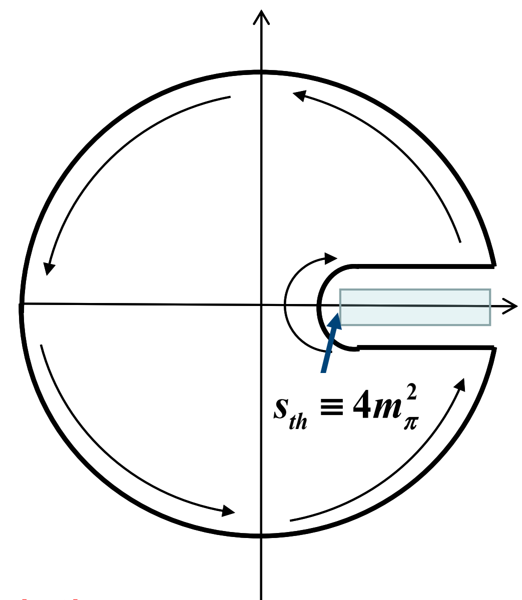
Anisovich & Anselm'66

1.4 Determination of the form factors : $F_\pi(s)$

- Cauchy Theorem: build the FF in the entire phase space

$$F(s) = \frac{1}{2i\pi} \oint_C \frac{F(s')}{(s'-s)} ds'$$

$$= \frac{1}{\pi} \int_{s_{th}}^{\Lambda^2} ds' \frac{disc(F(s))}{s'-s-i\varepsilon} + \frac{1}{2i\pi} \oint_{s=|\Lambda^2|} ds' \frac{F(s')}{s'-s}$$



$\Lambda \to \infty$

$$F(s) = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{disc\left[F(s')\right]}{s'-s-i\varepsilon} ds'$$

Dispersion Relation

4.4 Dispersion Relations for the $M_I(s)$

- $$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0^0(s')\,\hat{M}_0(s')}{|\Omega_0(s')|\left(s'-s-i\varepsilon\right)}\right)$$

  Omnès function

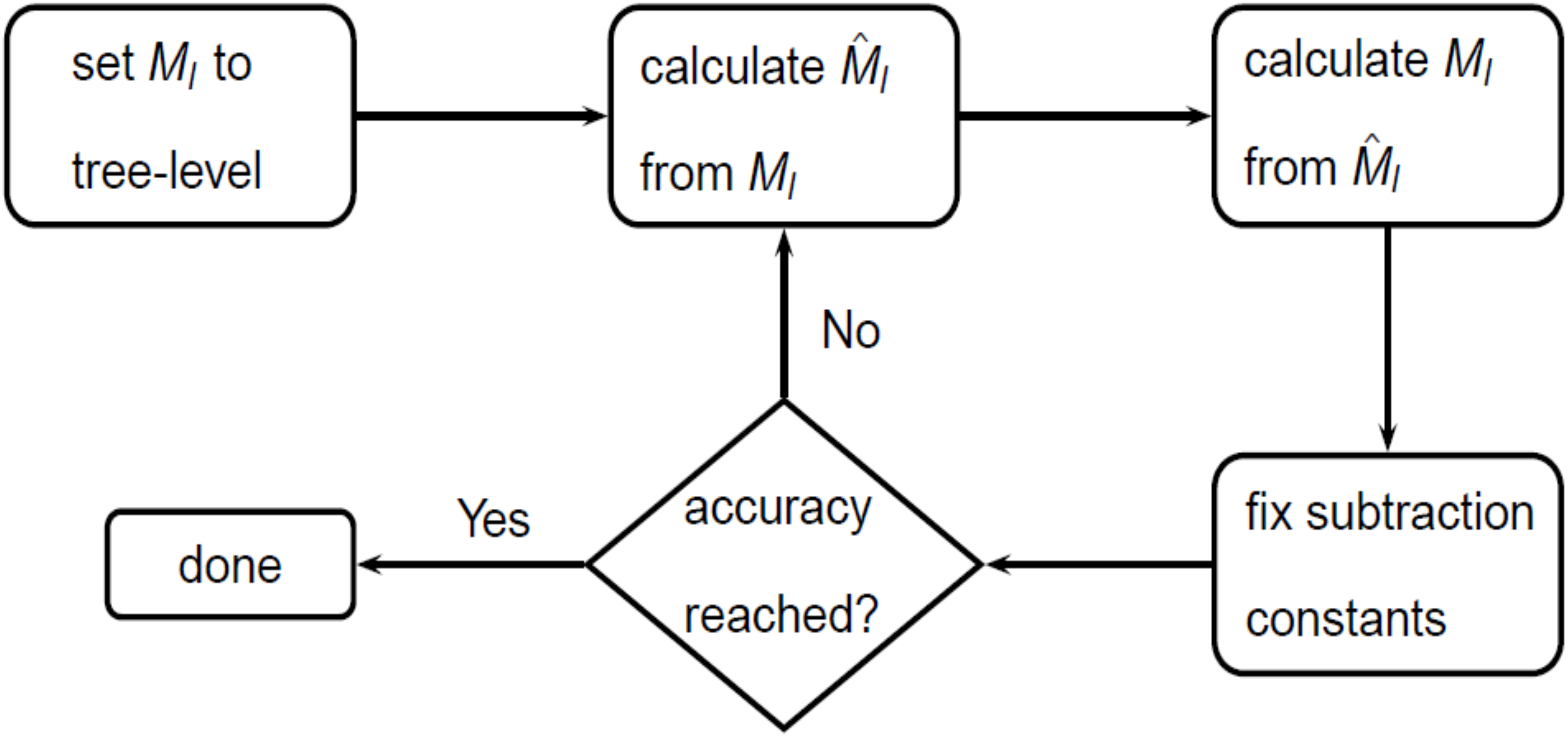
  Similarly for M_1 and M_2

- Four subtraction constants to be determined: α_0, β_0, γ_0 and one more in M_1 (β_1)

- Inputs needed for these and for the $\pi\pi$ phase shifts δ_ℓ^I
  - M_0: $\pi\pi$ scattering, ℓ=0, I=0
  - M_1: $\pi\pi$ scattering, ℓ=1, I=1
  - M_2: $\pi\pi$ scattering, ℓ=0, I=2

- Solve dispersion relations numerically by an iterative procedure

3.4 Iterative Procedure



3.5 Subtraction constants

- Extension of the numbers of parameters compared to *Anisovich & Leutwyler'96*

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$
$$P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2$$
$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2$$

- In the work of *Anisovich & Leutwyler'96* matching to one loop ChPT
  Use of the SU(2) x SU(2) chiral theorem
  ⇨ The amplitude has an *Adler zero* along the line s=u

- Now data on the Dalitz plot exist from KLOE, WASA and MAMI
  ⇨ Use the data to directly fit the subtraction constants

- Solution *linear* in the *subtraction constants* *Anisovich & Leutwyler'96*

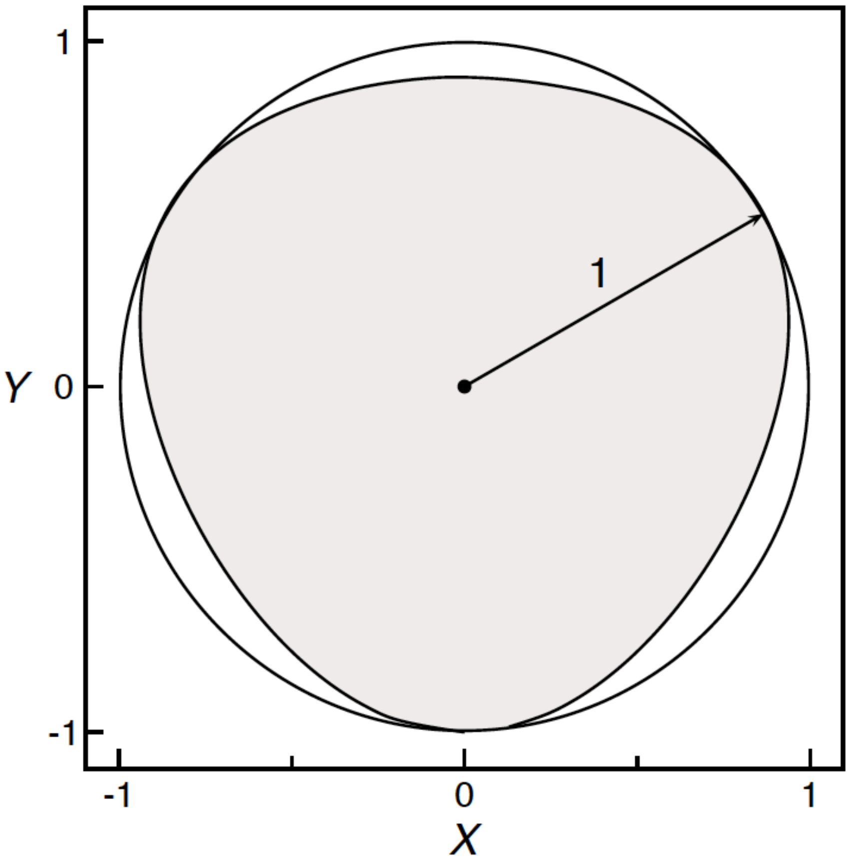
$$M(s,t,u) = \alpha_0 M_{\alpha_0}(s,t,u) + \beta_0 M_{\beta_0}(s,t,u) + \ldots$$

⇨ makes the fit much easier

Experimental measurements

- Dalitz plot measurement : Amplitude expanded in X and Y around X=Y=0

$$\left|A(s,t,u)\right|^2 = \Gamma(X,Y) = N\left(1 + aY + bY^2 + dX^2 + fY^3\right)$$



$$X = \frac{\sqrt{3}\,\left(T_+ - T_-\right)}{Q_c} = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$

$$Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

with T_i: kinetic energy of π^i in the η rest frame

and $Q_c \equiv T_0 - T_+ - T_- = M_\eta - 2M_{\pi^+} - M_{\pi^0}$

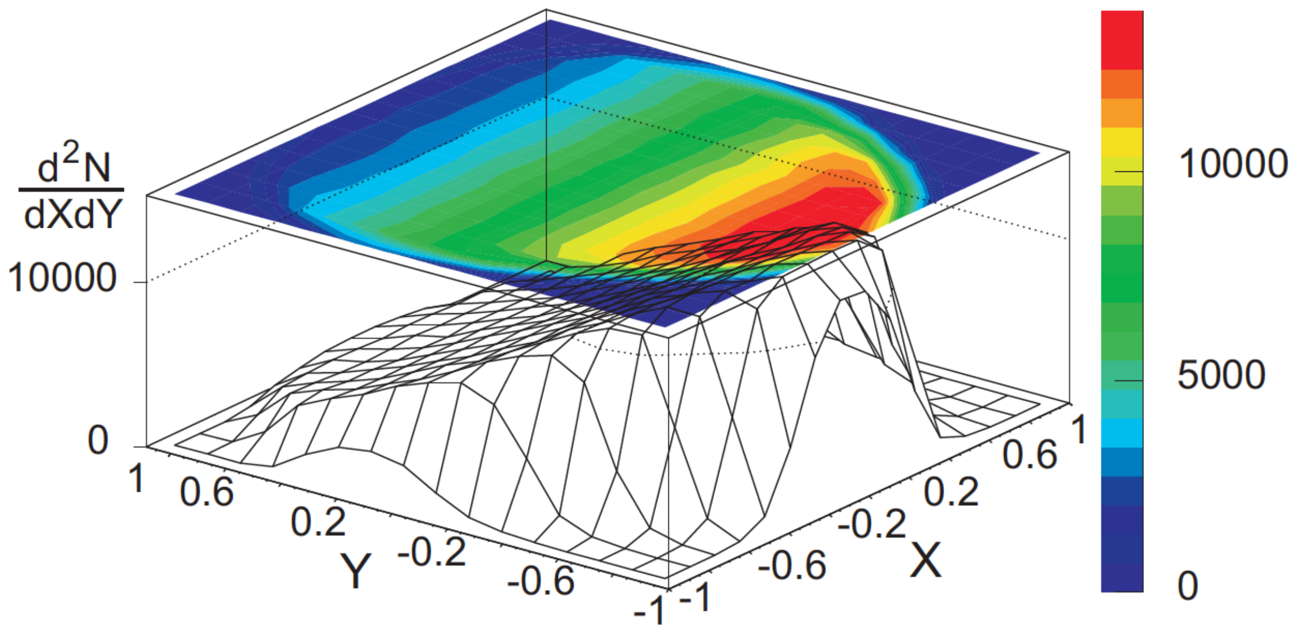
Experimental measurements : Charged channel

- Charged channel measurements with high statistics from *KLOE* and *WASA*

  e.g. *KLOE:* ~1.3 x 10^6 $\eta \rightarrow \pi^+ \pi^- \pi^0$ events from $e^+e^- \rightarrow \varphi \rightarrow \eta\gamma$

$$\left|A_c(s,t,u)\right|^2 = N\left(1 + aY + bY^2 + dX^2 + fY^3\right)$$

KLOE'08



$$Y = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

$$X = \frac{\sqrt{3}}{2M_\eta Q_c}(u - t)$$

Experimental measurements : Neutral channel

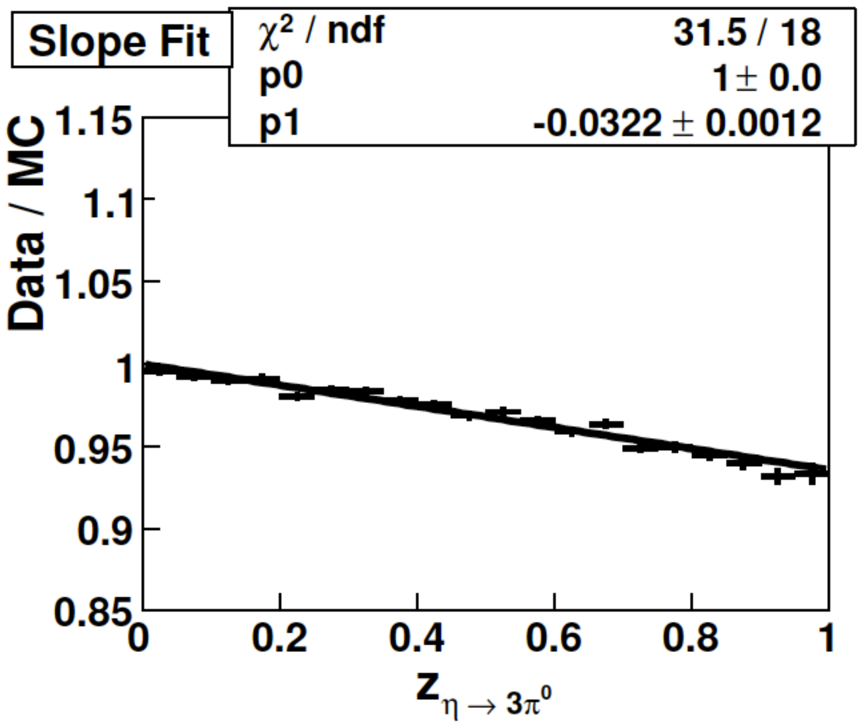
- Neutral channel measurements with high statistics from *MAMI-B*, *MAMI-C* and *WASA* e.g. *MAMI-C:* ~3 x 10^6 $\eta \to 3\pi^0$ events from $\gamma p \to \eta p$

$$\left|A_n(s,t,u)\right|^2 = N\left(1 + 2\alpha Z + 6\beta Y\left(X^2 - \frac{Y^2}{3}\right) + 2\gamma Z^2\right)$$

$$Z = \frac{2}{3}\sum_{i=1}^{3}\left(\frac{3T_i}{Q_n} - 1\right)^2 = X^2 + Y^2$$

$$Q_n \equiv M_\eta - 3M_{\pi^0}$$

⇨ Extraction of the slope :



MAMI-C'09

$$X = \frac{\sqrt{3}\,(T_+ - T_-)}{Q_c} = \frac{\sqrt{3}}{2M_\eta Q_c}(u - t)$$

$$Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

Experimental measurements : Neutral channel

- Neutral channel measurements with high statistics from *MAMI-B*, *MAMI-C* and *WASA* e.g. *WASA:* ~1.2 x 10^5 $\eta \to 3\pi^0$ events from $pp \to \eta pp$

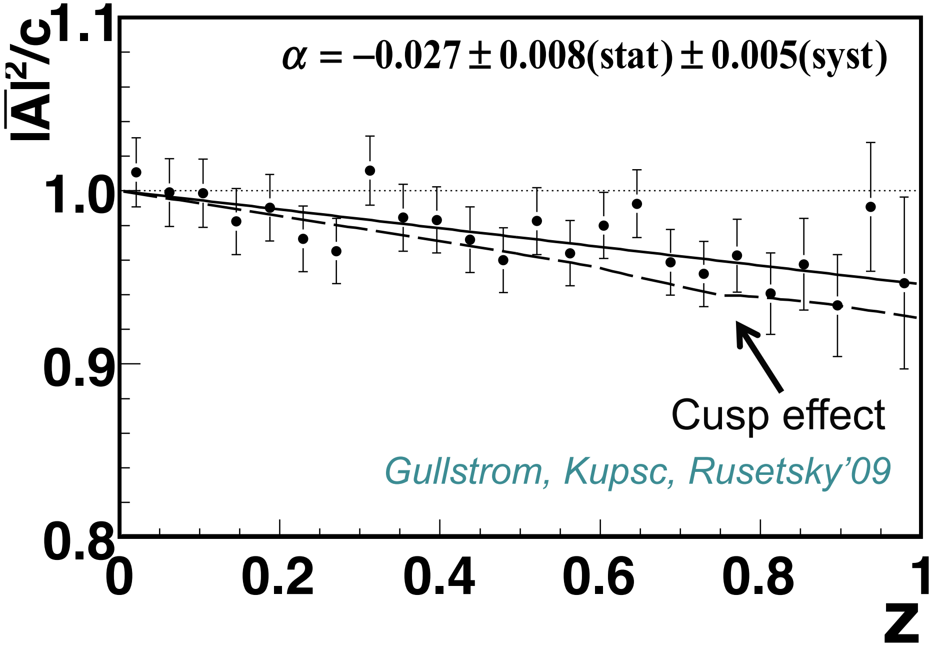
$$\left|A_n(s,t,u)\right|^2 = N\left(1 + 2\alpha Z + 6\beta Y\left(X^2 - \frac{Y^2}{3}\right) + 2\gamma Z^2\right)$$

$$Z = \frac{2}{3}\sum_{i=1}^{3}\left(\frac{3T_i}{Q_n} - 1\right)^2 = X^2 + Y^2$$

$$Q_n \equiv M_\eta - 3M_{\pi^0}$$

➡ Extraction of the slope :

WASA'09



$$X = \frac{\sqrt{3}\,(T_+ - T_-)}{Q_c} = \frac{\sqrt{3}}{2M_\eta Q_c}(u - t)$$

$$Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

3.4 Subtraction constants

- As we have seen, only Dalitz plots are measured, *unknown normalization!*

$$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M(s,t,u)$$

$$\left(Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \right)$$

To determine Q, one needs to know the normalization

⇨ For the normalization one needs to use ChPT

- The subtraction constants are

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$

$$P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2$$

$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2$$

Only *6 coefficients* are of physical relevance

3.4 Subtraction constants

- The subtraction constants are

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$

$$P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2$$

$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2$$

Only *6 coefficients* are of physical relevance

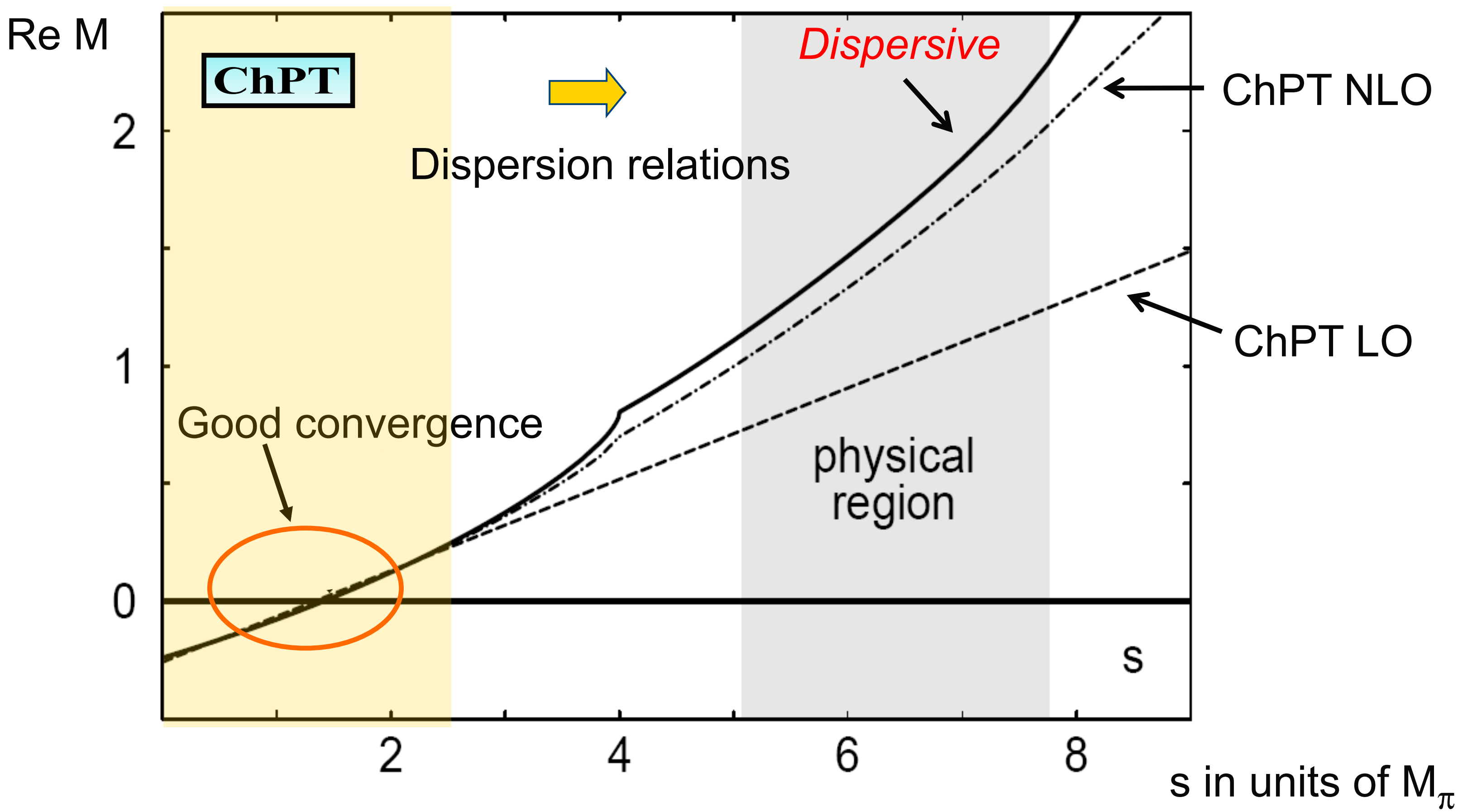
- They are determined from
  - Matching to one loop ChPT ⇨ $\delta_0 = \gamma_1 = 0$
  - Combine ChPT with fit to the data ⇨ δ_0 and γ_1 are determined from the data

- Matching to one loop ChPT : Taylor expand the dispersive M_I
  Subtraction constants ⇔ Taylor coefficients

Dispersive approach

- Dispersion Relations: extrapolate ChPT at higher energies

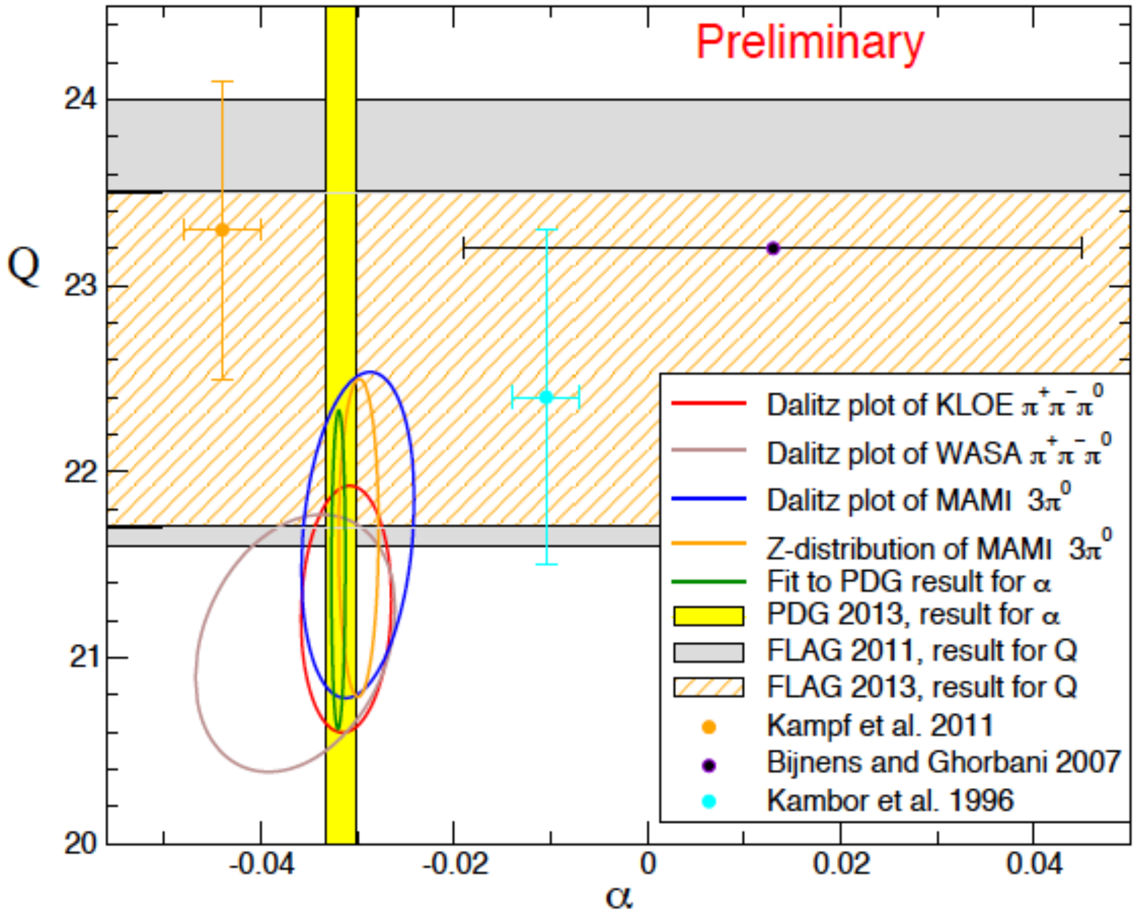
Anisovich & Leutwyler'96



- Important corrections in the physical region taken care of by the *dispersive treatment!*

4.3 Qualitative results of our analysis

- Plot of Q versus α :



NB: Isospin breaking has not been accounted for

From kaon mass spliting :

$$Q = 20.7 \pm 1.2$$

Kastner & Neufeld'08

- All the data give consistent results. The preliminary outcome for Q is intermediate between the lattice result and the one of Kastner and Neufeld.

$\eta \to 3\pi$

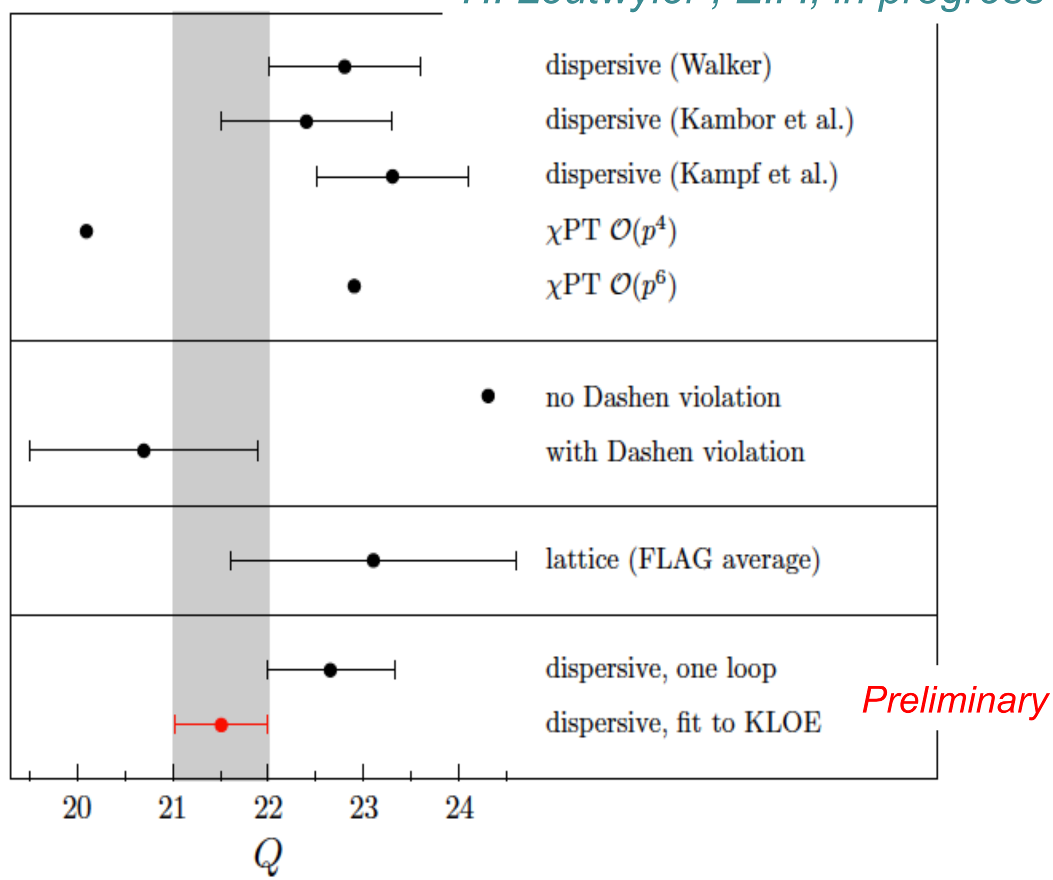
- Isospin violating process ⇨ possibility to extract the quark mass ratio Q:

G. Colangelo, S. Lanz, H. Leutwyler , E.P., in progress

$$\Gamma_{\eta\to 3\pi} \propto \int |A(s,t,u)|^2 \propto Q^{-4}$$

$$Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \qquad \left[\hat{m} \equiv \frac{m_d + m_u}{2}\right]$$

$$A(s,t,u) = \frac{N}{Q^2} M(s,t,u)$$



- M(s,t,u) determined through the dispersive analysis of the data but for N one has to rely on ChPT

- Analysis for JPAC by *P. Guo, I. Danilkin, D. Schott et al'15* using *WASA* data

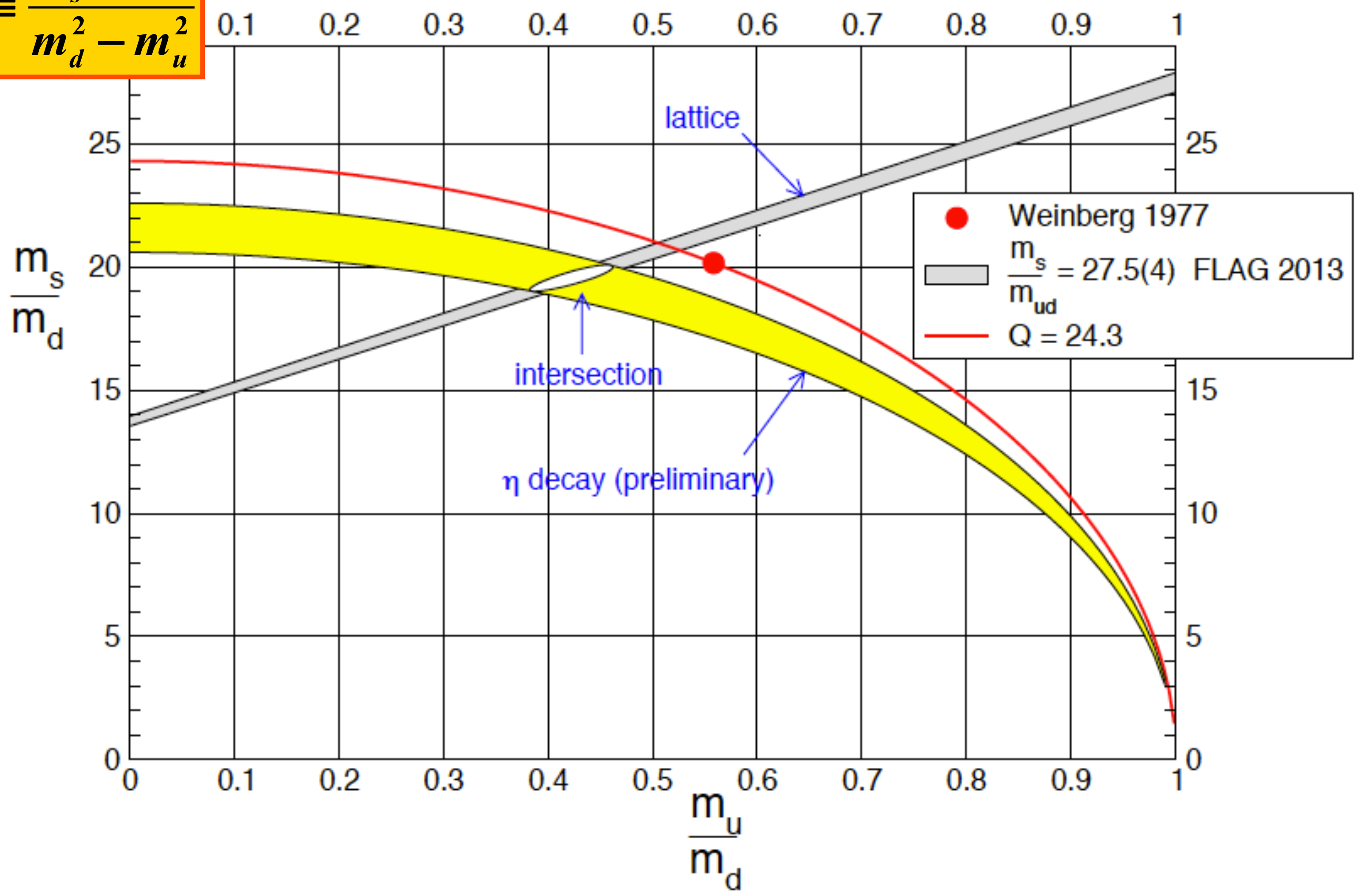
$$Q = 21.4 \pm 0.4$$

⇨ Analysis of *CLAS* data

2.4 Results: quark mass ratios

$$Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$$

H.Leutwyler

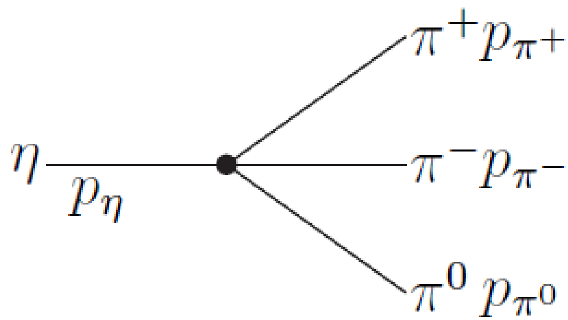


5. Back-up

1.6 Chiral expansion

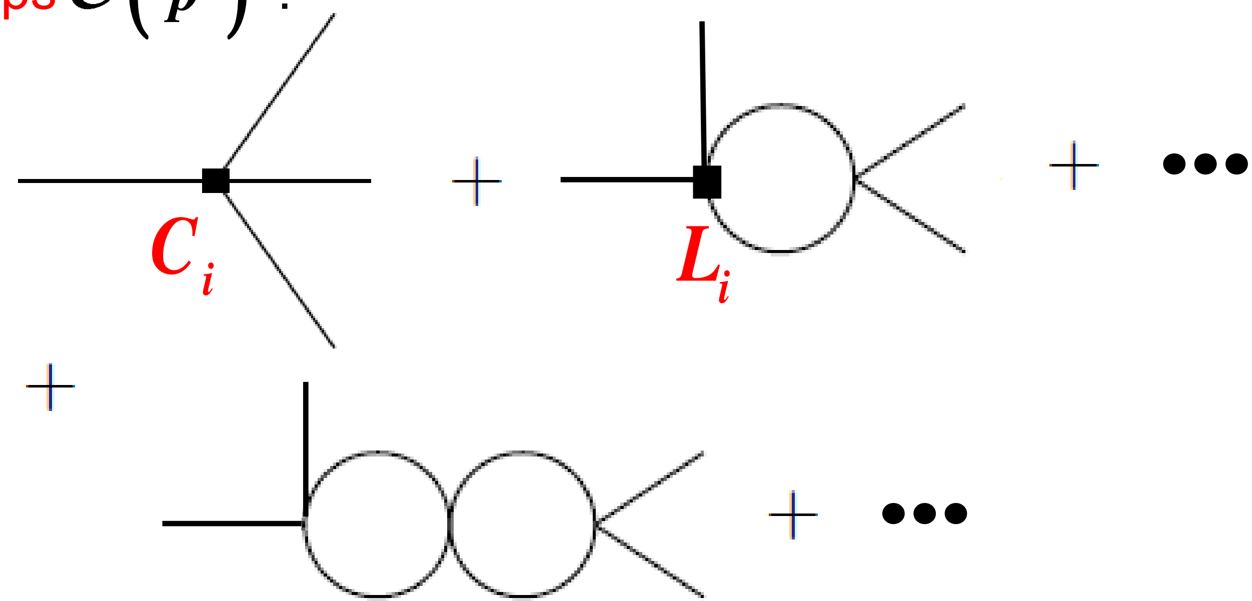
- Ex : $\eta \rightarrow \pi^+ \pi^- \pi^0$ ⇨ $A = A_2 + A_4 + A_6 + \ldots$

  - Tree level $\mathcal{O}\left(p^2\right)$:



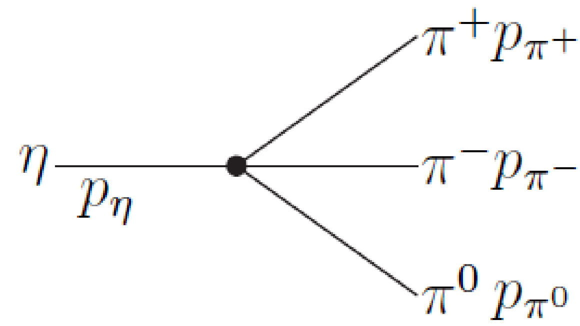
A_2

  - Two loops $\mathcal{O}\left(p^6\right)$:



A_6

2.2 Extraction of Q



- Extraction of the quark masses:

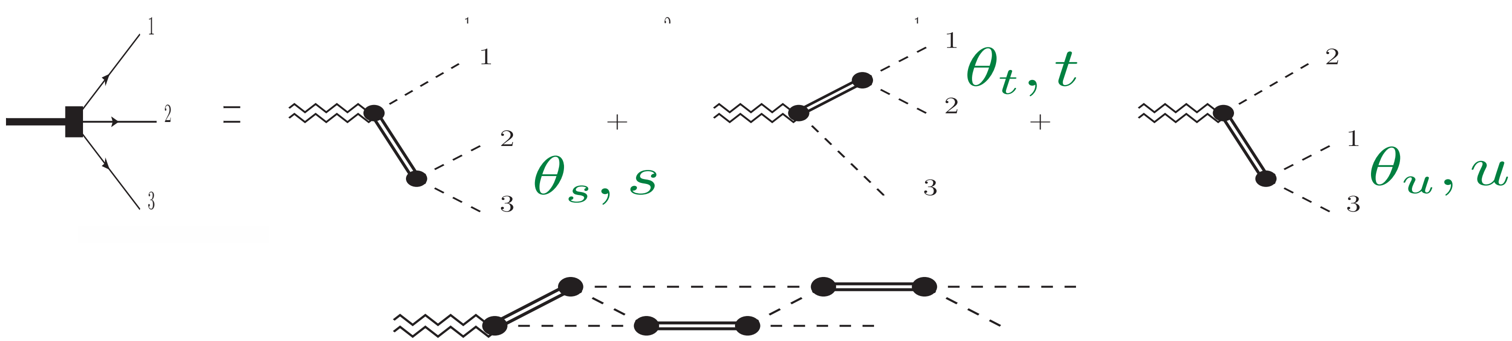
$$\Gamma_{\eta\to 3\pi} \propto Q^{-4} |M|^2 \quad \Rightarrow \quad Q^2 \propto (m_u - m_d)$$

Experiment ($\Gamma_{\eta\to 3\pi}$)

KLOE (Italy),
MAMI (Germany),
WASA (Sweden, Germany),
CLAS (JLab, USA)

Computed with dispersive methods + ChPT ($|M|^2$)

- Dispersive method: Take into account the *$\pi\pi$ final state interactions*



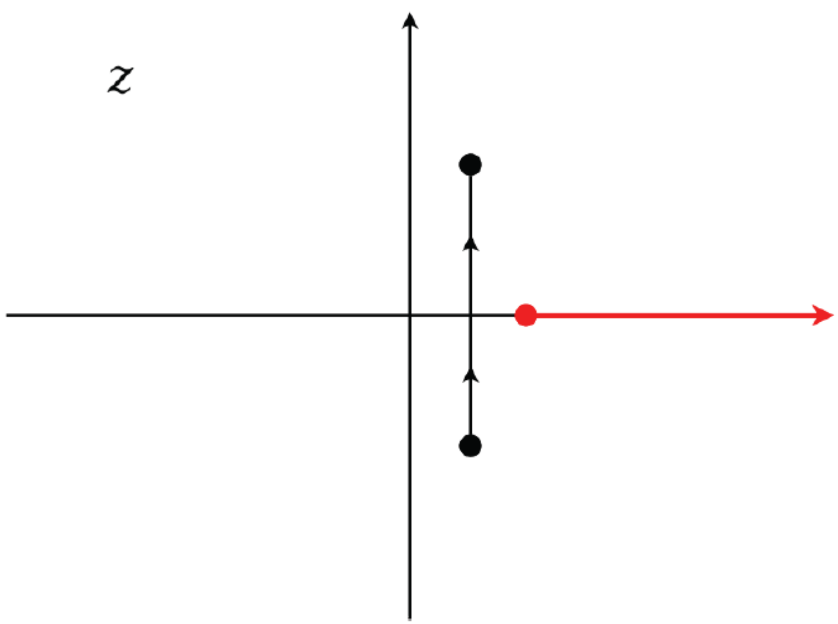
Discontinuities of the $M_I(s)$

- Ex: $\hat{M}_0(s) = \frac{2}{3}\langle M_0 \rangle + 2(s - s_0)\langle M_1 \rangle + \frac{20}{9}\langle M_2 \rangle + \frac{2}{3}\kappa(s)\langle z M_1 \rangle$

  where $\langle z^n M_I \rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z)),\; z = \cos\theta$ scattering angle

  Non trivial angular averages ⇨ need to deform the integration path to avoid crossing cuts

  Anisovich & Anselm'66



Discontinuities of the $M_I(s)$

- Ex: $\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle + 2(s-s_0)\langle M_1\rangle + \frac{20}{9}\langle M_2\rangle + \frac{2}{3}\kappa(s)\langle zM_1\rangle$

  where $\langle z^n M_I\rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z)),\ z = \cos\theta$ scattering angle

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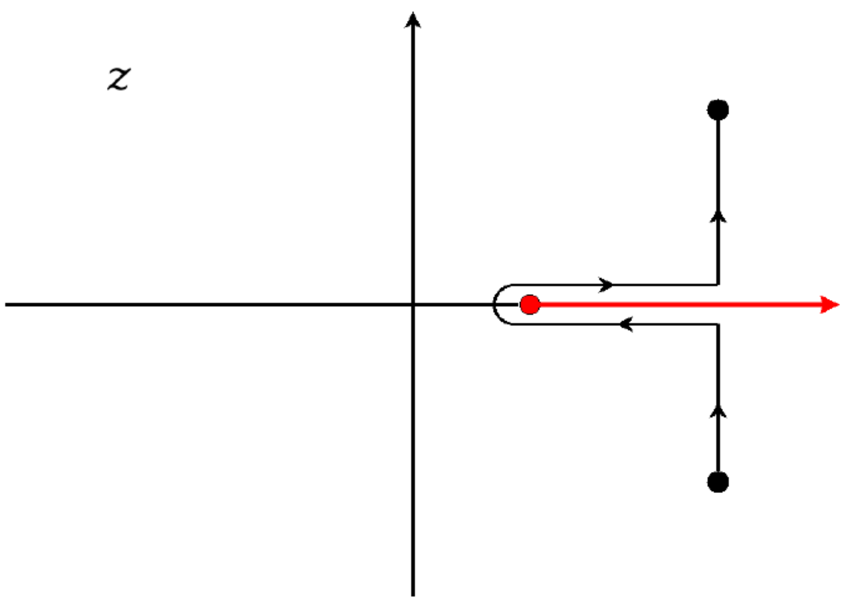
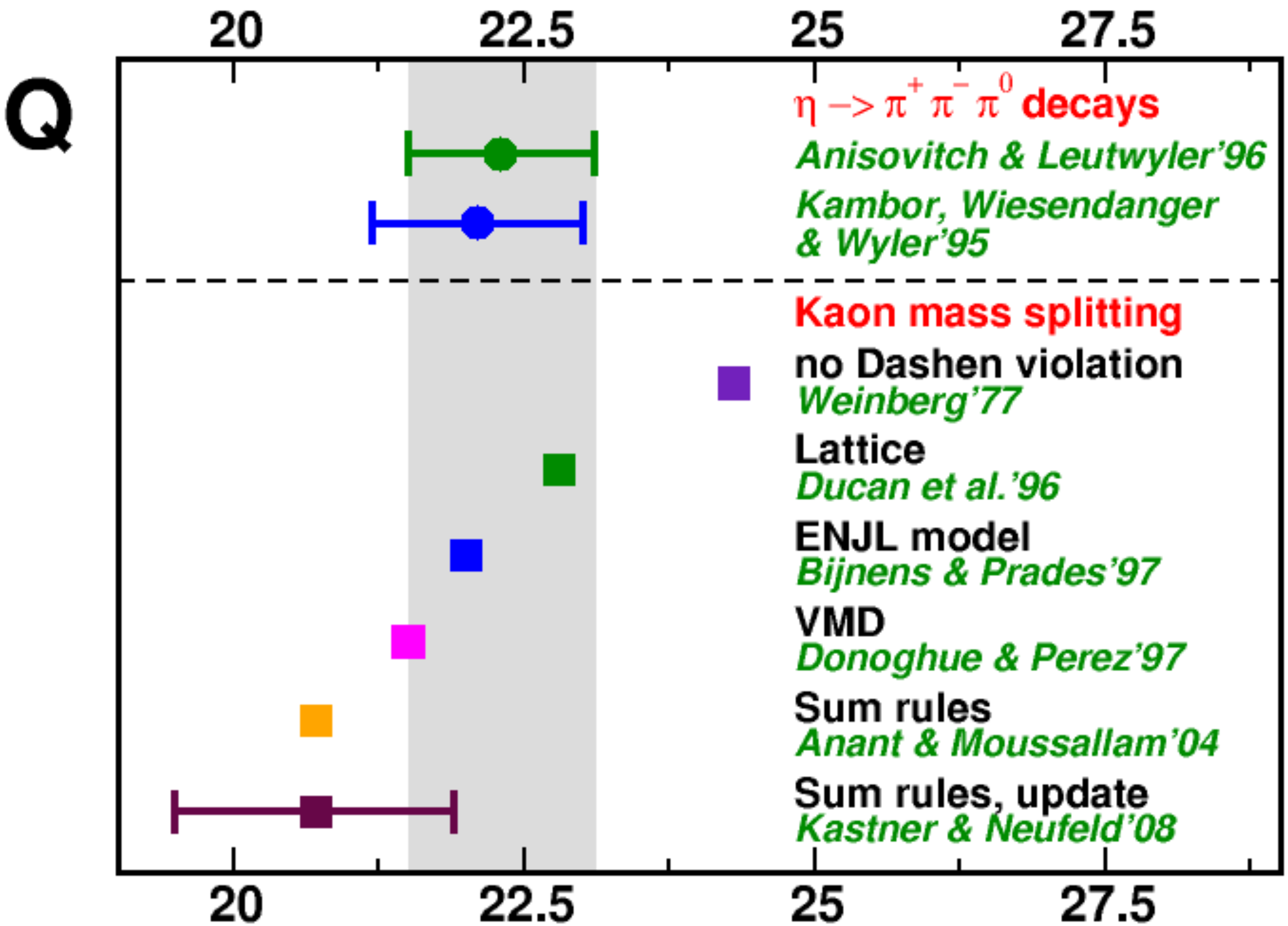


3.7 Comparison of values of Q



$Q = 22.3 \pm 0.9$

$Q = 22.1 \pm 0.9$

$Q = 24.2$

$Q = 22.8$

$Q = 22$

$Q = 21.5$

$Q = 20.7$

$Q = 20.7 \pm 1.2$

➡ Fair agreement with the determination from meson masses