I.] Extracting the quark mass ratio from $\eta \to 3\pi$

The Lagrangian describing strong interactions:

$$\mathcal{L}_{QCD} = -\frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} + \sum_{k=1}^{N_f} \bar{q}_k (i\gamma^\mu D_\mu - m_k) q_k$$

$G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \ldots$ $\qquad D_\mu = \partial_\mu - i g_s \frac{\lambda_a}{2} A^a_\mu(x)$ ← gluon field

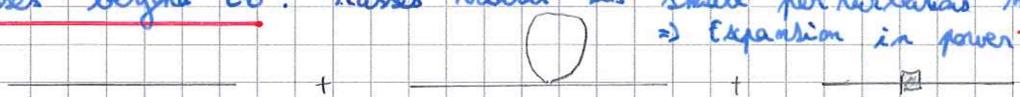
- Build all the invariants under $SU(3)_c$ with the quarks
- gauge the theory: $SU(3)_c \to$ local $\underline{q_k}$

  8 different gauge fields: G^a_μ the gluons

- Different parts of the Lagrangian:
  - kinetic terms $-\frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu}$
  - Interactions quark - gluon
  - Interactions gluon - gluon: non - abelian gauge group.
- One single universal coupling: $\alpha_s(\mu) = \frac{g_s^2(\mu)}{4\pi}$ strong coupling constant, depends on energy.
- 7 unknowns in the Lagrangian
  - * strong coupling constant: $\alpha_s(\mu) = \frac{g_s^2(\mu)}{4\pi}$
  - * 6 quark masses: m_k
- How to get the quark masses?

→ No direct access due to confinement!

Masses beyond LO: Masses treated as small perturbations $m_{u,d,s} \ll \Lambda_{QCD}$
⇒ Expansion in power of m_q



$$\left.\begin{aligned} M^2_{\pi^+} &= B_0 (m_u + m_d) \{1 + \mathcal{O}(\hat{m}, m_s)\} \\ M^2_{K^+} &= B_0 (m_u + m_s) \{1 + \mathcal{O}(\hat{m}, m_s)\} \\ M^2_{K^0} &= B_0 (m_d + m_s) \{1 + \mathcal{O}(\hat{m}, m_s)\} \end{aligned}\right\}$$

$\boxed{\text{GMOR relation:}}$

$\left(\begin{matrix}\text{Meson}\\ \text{mass}\end{matrix}\right)^2 =$ Spontaneous sym breaking ChSB $\langle \bar{q}q \rangle$ × explicit sym breaking ChSB m_q

Gell-Mann - Oakes - Renner relations:

Gell Mann Oakes Renner formula 1968 Leutwyler's lecture

$$\boxed{M_\pi^2 = \underset{\substack{\uparrow \\ \text{explicit}}}{(m_u + m_d)} \cdot \underset{\substack{\uparrow \\ \text{spontaneous.}}}{\langle 0|\bar{u}u|0\rangle} \times \frac{1}{F_\pi^2}}$$

Coefficient : decay constant F_π

- why $M_\pi^2 \propto (m_u + m_d)$?

$\langle 0 | \bar{u}(x) \gamma^\mu \gamma_5 d(x) | \pi^- \rangle = i\sqrt{2}\, F_\pi\, p^\mu e^{-ipx}$ → Axial current generates pion.

$\langle 0 | \bar{u}(x)\, i\gamma_5\, d(x) | \pi^- \rangle = \sqrt{2}\, G_\pi\, e^{-ipx}$

Current conservation : Use Dirac equation → see following page

Exercise → $\partial_\mu (\bar{u}\gamma^\mu\gamma_5 d) = (m_u$

$= \partial_\mu \bar{u}\, \gamma^\mu \gamma_5 d + \bar{u}\gamma^\mu \gamma_5 \partial_\mu d$

$= i m_u\, \bar{u}\gamma^\mu\gamma_5 d - \underbrace{\bar{u}\,\gamma_5 \gamma^\mu \partial_\mu d}$

$+ i\bar{u}\, \gamma_5\, m_d\, d$

$= (m_u + m_d)\, \bar{u}\, i\gamma_5\, d$

$\boxed{x = 0}$

$-i^2 \sqrt{2}\, F_\pi\, p^\mu p_\mu = (m_u + m_d)\sqrt{2}\, G_\pi$

$\sqrt{2}\, F_\pi\, p^2 = (m_u + m_d)\sqrt{2}\, G_\pi$

$F_\pi\, M_\pi^2 = (m_u + m_d)\, G_\pi$

$\boxed{\gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5}$

$$\boxed{M_\pi^2 = (m_u + m_d)\frac{G_\pi}{F_\pi}}$$ exact

- Expansion in powers of m_u, m_d:

$\frac{G_\pi}{F_\pi} = B + O(m)$

$$\Rightarrow \boxed{M_\pi^2 = (m_u + m_d)\, B + O(m^2)}$$

'1.b'

~~C = iγ^0~~

$\psi \to -i\gamma_2 \psi^* = \psi_c$

$\bar{\psi} = \psi^\dagger \gamma^0$

$C \gamma^\mu C^{-1} = -\gamma^{\mu *}$

$(i\partial\!\!\!/ - m)\psi = 0$

$C: \ (i\partial\!\!\!/ - m)\psi = 0$

$C(i\partial\!\!\!/ - m)\psi = 0$

$C(i\partial\!\!\!/ - m)C^{-1}C\psi = 0$

$(C i\partial\!\!\!/ C^{-1} - m)(-i\gamma_2\psi^*) = 0$

$(i(-\partial\!\!\!/) - m)(-i\gamma_2\psi^*) = 0$

$-(i\partial\!\!\!/ - m)\psi_c = 0$

$i \to -i$ oder C? $\Rightarrow (i\partial\!\!\!/ + m)\psi_c = 0$

$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$

$[(i\partial\!\!\!/ - m)\psi]^\dagger = 0$

$\psi^\dagger (i\partial\!\!\!/ - m)^\dagger = 0$

$\psi^\dagger(-i(\gamma^0\gamma^\mu\gamma^0)\partial - m) = 0$

$\psi^\dagger \gamma^0 (-i\gamma^0\gamma^0\gamma^\mu\gamma^0\partial - \gamma^0\gamma^0 m) = 0$

$\psi^\dagger\gamma^0(-i\gamma^\mu\gamma^0\partial - \gamma^0 m) = 0$

$\bar{\psi}(-i\gamma^\mu\partial_\mu - m)\gamma^0 = 0$

$\bar{\psi}(i\gamma^\mu\partial_\mu + m) = 0$

$\rightarrow \boxed{\bar{\psi}(i\partial\!\!\!/ + m) = 0}$

$(i\not{\partial} - m_d)d$

$$\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = (\partial_\mu\bar{u})\gamma^\mu\gamma_5 d + \bar{u}\gamma^\mu\gamma_5\partial_\mu d$$

$$= \bar{u}\overleftarrow{\not{\partial}}\gamma_5 d + \bar{u}(\not{\partial}\gamma_5 d)$$

$$= \bar{u}\overleftarrow{\not{\partial}}\gamma_5 d - \bar{u}\gamma_5\not{\partial} d \quad \not= 0$$

$\bar{u}\overleftarrow{\not{\partial}}$

$$= \bar{u}(im_u)\gamma_5 d + \bar{u}\gamma_5 i m_d d .$$

$$\cancel{= i(\bar{u}\, m_u\gamma_5}$$

$$= i\left[m_u\,\bar{u}\gamma_5 d + m_d\,\bar{u}\gamma_5 d\right]$$

Electromagnetic effects : Dashen's theorem : 2/

$(M^2_{K^+} - M^2_{K^0})_{em} - (M^2_{\pi^+} - M^2_{\pi^0})_{em} = O(e^2 m)$

s quark d quark same charge

ChPT at leading order + e.m. corrections :

$M^2_{\pi^0} = B_0(m_u + m_d)$, $M^2_{\pi^+} = B_0(m_u + m_d) + \Delta_{em}$

$M^2_{K^0} = B_0(m_d + m_s)$, $M^2_{K^+} = B_0(m_u + m_s) + \Delta_{em}$

$m_\eta = 547.862$ MeV

2 unknowns : B_0 and Δ_{em} .

Exercise : get $\frac{m_u}{m_d}$ and $\frac{m_s}{m_d}$ from meson masses

Answer $\frac{m_u}{m_d} = \ldots$ Weinberg '77

PDG '14 $M^2_{\pi^+} = 0.13957018$ GeV

$M^2_{\pi^0} = 0.1349766$ GeV

$M^2_{K^+} = 0.493677$ GeV

$M^2_{K^0} = 0.497614$ GeV

$$\frac{m_u}{m_d} \overset{L.O.}{=} \frac{M^2_{K^+} - M^2_{K^0} + 2M^2_{\pi^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} \sim 0.56$$

$$\frac{m_s}{m_d} \overset{L.O.}{=} \frac{M^2_{K^+} + M^2_{K^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} \sim 20.2$$

→ $\boxed{\text{ChPT}}$ what is it ? → see new page

- Form dimensionless ratios : Mass formulae to second chiral order :

$$\left(\frac{M_K^2}{M_\pi^2}\right)_{st} = \frac{\frac{1}{2}B_0(M^2_{K^+} + M^2_{K^0})}{\frac{1}{2}(M^2_{\pi^+} + M^2_{\pi^0})} = \frac{\frac{1}{2}[B_0(m_u + m_s) + B_0(m_d + m_s)]}{\frac{1}{2}\cdot[B_0(m_u + m_d) + B_0(m_u + m_d)]}(1 + \Delta_M + O(m_q^2))$$

$$= \frac{B_0(m_s + \frac{m_u + m_d}{2})}{B_0(2\,\frac{m_u + m_d}{2})}(1 + \Delta_M + O(m_q^2))$$

[Gasser & Leutwyler '85]

$$= \frac{m_s + \hat{m}}{2\hat{m}}(1 + \Delta_M + O(m_q^2))$$

$$\frac{(M^2_{K^0} - M^2_{K^+})_{st}}{M^2_K - M^2_\pi} = \frac{B_0(m_d - m_u)}{B_0(m_s + \hat{m}) - 2B_0\hat{m}}(1 + \Delta_M + O(m_q^2))$$

$$= \frac{m_d - m_u}{m_s - \hat{m}}(1 + \Delta_M + O(m_q^2))$$

with $\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F^2}(2L_8 - L_5) + \chi$ Logs

- Form double ratio : $Q^2 = \frac{M_K^2}{M_\pi^2} \cdot \frac{M_K^2 - M_\pi^2}{(M^2_{K^0} - M^2_{K^+})_{st}}(1 + O(m_q^2, e^2))$

no corrections at one loop

Corrections suppressed by 2 orders $\hat{m}^2$ quark masses expansion

$$= \frac{m_s + \hat{m}}{2\hat{m}} \cdot \frac{m_s - \hat{m}}{m_d - m_u} = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$$

- Form dimensionless ratio : Mass formula to second chiral order — 2.b

Gasser & Leutwyler '85

Exercise : derive LO (first order

$$\left(\frac{M_K^2}{M_\pi^2}\right)_{str} = \frac{\frac{1}{2}\left(M^2_{K^+} + M^2_{K^0}\right)}{\frac{1}{2}\left(M^2_{\pi^+} + M^2_{\pi^0}\right)}\left(\underset{LO}{1} + \underset{NLO}{\Delta_M} + O(m_q^2)\right)$$

$$= \frac{B_0\left(m_s + \frac{m_u + m_d}{2}\right)}{B_0\left(2\,\frac{m_u + m_d}{2}\right)}\left(1 + \Delta_M + O(m_q^2)\right)$$

$$\left(\frac{M_K^2}{M_\pi^2}\right)_{str} = \frac{m_s + \hat{m}}{2\hat{m}}\left(1 + \Delta_M + O(m_q^2)\right)$$

$$\frac{\left(M^2_{K^0} - M^2_{K^+}\right)_{str}}{\left(M_K^2 - M_\pi^2\right)_{str}} = \frac{B_0\left(m_d - m_u\right)}{B_0\left(m_s + \hat{m}\right) - 2B_0\hat{m}}\left(1 + \Delta_M + O(m_q^2)\right)$$

$$= \frac{m_d - m_u}{m_s - \hat{m}}\left(1 + \Delta_M + O(m_q^2)\right)$$

with $\boxed{\Delta_M = \frac{8\left(M_K^2 - M_\pi^2\right)}{F_0^2}\left(2L_8 - L_5\right) + \chi\ \text{logs}}$

$\hookrightarrow \frac{1}{32\pi^2 F_0^2}\left(m_\eta^2 \log\frac{m_\eta^2}{\mu} - m_\pi^2 \log\frac{m}{\mu}\right.$

- Form double ratio: Δ_M cancels in the ratio

$$\frac{\left(M^2_{K^0} - M^2_{K^+}\right)_{str}}{\left(M_K^2 - M_\pi^2\right)_{str}} \cdot \frac{1}{\left(\frac{M_K^2}{M_\pi^2}\right)_{str}} = \frac{m_d - m_u}{m_s - \hat{m}}\left(1 + \cancel{\Delta_M}\right)^{O(m^2)} \cdot \frac{2\hat{m}}{\left(m_s + \hat{m}\right)} \cdot \frac{1}{\left(1 + \cancel{\Delta}\right.}$$

$$= \frac{\left(m_d - m_u\right)\left(m_d + m_u\right)}{}$$

$$\left(\frac{M_K^2}{M_\pi^2}\right)_{str} \cdot \left.\frac{M_K^2 - M_\pi^2}{\left(M^2_{K^0} - M^2_{K^+}\right)}\right|_{str} = \frac{m_s + \hat{m}}{2\hat{m}}\left(1 + \Delta_M + O(m_q^2)\right) \cdot \frac{m_s - \hat{m}}{m_d - m_u}\,\frac{1}{1 + \Delta}$$

$$= \underbrace{\boxed{\frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}}}_{Q^2}\left(1 + O(m_q^2)\right)$$

<u>Exercise</u> : Show that from $Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$ one can draw an ellipse called "Leutwyler Ellipse" in the plane $\frac{m_s}{m_d} = f\left(\frac{m_u}{m_d}\right)$ up to $\left(\frac{\hat{m}}{m_s}\right)^2$ corrections
neglecting "

Draw the ellipse and put Q_D : value of Q relying on meson masses + the use of Dashen's theorem. [Hint: take $(M^2_{\pi^+} - M^2_{\pi^0})_{em} = (M^2_{\pi^+} - M^2_{\pi^0})_{phys}$]

<u>Answer</u>:

$$\frac{1}{Q^2}\,\frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = 1$$

$$\frac{1}{Q^2}\,\frac{m_s^2}{m_d^2} \cdot \frac{1 - \frac{\hat{m}^2}{m_s^2}}{1 - \frac{m_u^2}{m_d^2}} = 1$$

$$\frac{1}{Q^2}\,\frac{m_s^2}{m_d^2}\left(1 - \frac{\hat{m}^2}{m_s^2}\right) = 1 - \frac{m_u^2}{m_d^2}$$

$$\frac{1}{Q^2}\,\frac{m_s^2}{m_d^2}\left(1 - \frac{\hat{m}^2}{m_s^2}\right) + \frac{m_u^2}{m_d^2} = 1$$

Neglecting $\frac{\hat{m}^2}{m_s^2}$ that is tiny we get:

$$\boxed{\frac{m_u^2}{m_d^2} + \frac{1}{Q^2}\,\frac{m_s^2}{m_d^2} = 1}$$: equation of "Leutwyler's ellipse"

where Q represents the major semi-axis of the ellipse.

Compute Q_{Dashen}: relies on meson masses + Dashen theorem:

$$(M^2_{K^0} - M^2_{K^+})_{str} = (M^2_{K^0} - M^2_{K^+}) - (M^2_{K^0} - M^2_{K^+})_{em}$$

$$\underset{\text{Dashen}}{=} (M^2_{K^0} - M^2_{K^+}) - (M^2_{\pi^0} - M^2_{\pi^+})_{em}$$

$$= (M^2_{K^0} - M^2_{K^+}) - (M^2_{\pi^0} - M^2_{\pi^+})_{phys} \rightarrow -(M^2_{\pi^+} - M^2_{\pi^0} + 2M^2_{K^0})_{em}$$

$$Q_D^2 = \left(\frac{M_K^2}{M_\pi^2}\right)_{str} \cdot \frac{(M_K^2 - M_\pi^2)_{str}}{(M^2_{K^0} - M^2_{K^+})_{str}} = \frac{\frac{1}{2}(M^2_{K^+} + M^2_{K^0}) - \frac{1}{2}(M^2_{K^+} + M^2_{K^0})_{em} \cdot \left(\frac{1}{2}(M^2_{K^+} + M^2_{K^0}) - (M^2_{\pi^+} + M^2_{\pi^0})\right)}{\frac{1}{2}(M^2_{\pi^+} + M^2_{\pi^0}) - \cancel{\frac{1}{2}(M^2_{\pi^+} + M^2_{\pi^0})_{em}} \cdot (M^2_{K^0} - M^2_{K^+} - M^2_{\pi^0} + M^2_{\pi^+})}$$

$$\Rightarrow \quad Q_D = 24.2$$

<u>But</u> various calculations find <u>large corrections</u> to Dashen's theorem

$O(e^2 m_q)$ → m_s!

$$= \frac{(M^2_{K^+} + M^2_{K^0} - M^2_{\pi^+} + M^2_{\pi^0})(M^2_{K^+} + M^2_{K^0} - M^2_{\pi^+} - M^2_{\pi^0})}{4\,M^2_{\pi^0}\,(M^2_{K^0} - M^2_{K^+} - M^2_{\pi^0} + M^2_{\pi^+})}$$

$$1 \leq \frac{(M_{K^+}^2 - M_{K^0}^2)_{e.m}}{(M_{\pi^+}^2 - M_{\pi^0}^2)_{e.m}} \leq 2.5 \; ! \quad \Rightarrow \quad 20.6 \leq Q \leq 24.2 \Rightarrow \text{plot}$$

Not very precise!

. Can we extract Q with a better precision? → lattice QCD

Yes! using $\eta \to 3\pi$ decays! η PDG

But how?

→ see slide

. $\eta \to 3\pi$: η has $I = 0 \rightsquigarrow G = +$

π has $I = 1 \rightsquigarrow G = -$

Ex: → By Charge conservation give η decay.

→ violates G parity / isospin conservation

η : spin zero isospin singlet state

→ 3π must couple to vanishing isospin and angular momentum.

⇒ Impossible because that the $|0,0\rangle$ state formed by 3π can not have angular momentum 0

<u>Exercise</u>: Show that the state $|0,0\rangle$ formed by 3π can not have a zero angular momentum.

<u>Physics basis</u>: I, I_3

$|\pi^+\rangle = |1,+1\rangle$, $|1,0\rangle = |\pi^0\rangle$, $|1,-1\rangle = |\pi^-\rangle$

<u>Isospin basis</u>

$|\pi^\pm\rangle = \frac{1}{\sqrt{2}}(|\pi^1\rangle \pm i|\pi^2\rangle)$

$|\pi^0\rangle = |\pi^3\rangle$

A generic 2 pion state has no definite isospin but it can be decomposed in a sum of egeinstates of I and I_3

From 3 pion flavours one can construct 9 different 2 pion states and their decomposition is given by: 2 pions form eigenstates of I_3 because the third component can simply be added!

$$\left.\begin{aligned} |\pi^+\rangle \otimes |\pi^+\rangle &= |1,1\rangle \otimes |1,1\rangle \\ &= |2,2\rangle \\ &= |\pi^+,\pi^+\rangle \end{aligned}\right\} \Rightarrow |\pi^+,\pi^+\rangle = \underset{\substack{\uparrow \\ 1.}}{C}|2,2\rangle$$

$$\left.\begin{aligned} |\pi^-\rangle \otimes |\pi^-\rangle &= |1,-1\rangle \otimes |1,-1\rangle \\ &= |2,-2\rangle \\ &= |\pi^+,\pi^-\rangle \end{aligned}\right\} \quad |\pi^-,\pi^-\rangle = \underset{\substack{\uparrow \\ 1}}{C}\;|2,-2\rangle$$

$$|\pi^0\rangle \otimes |\pi^0\rangle = \sqrt{\tfrac{2}{3}}\,|2,0\rangle - \tfrac{1}{\sqrt{3}}\,|0,0\rangle$$
$$|\pi^0\pi^0\rangle$$

$$|\pi^+\rangle \otimes |\pi^0\rangle = |1,+1\rangle \otimes |1,0\rangle$$
$$= \tfrac{1}{\sqrt{2}}\,|2,1\rangle + \tfrac{1}{\sqrt{2}}\,|1,1\rangle$$

$$|\pi^-\rangle \otimes |\pi^0\rangle = |1,-1\rangle \otimes |1,0\rangle$$
$$|\pi^-\pi^0\rangle = \tfrac{1}{\sqrt{2}}\,|2,-1\rangle - \tfrac{1}{\sqrt{2}}\,|1,-1\rangle$$

$$|\pi^0\rangle \otimes |\pi^+\rangle = |1,0\rangle \otimes |1,1\rangle$$
$$|\pi^0\pi^+\rangle = \tfrac{1}{\sqrt{2}}\,|2,1\rangle - \tfrac{1}{\sqrt{2}}\,|1,1\rangle$$

$$|\pi^0\rangle \otimes |\pi^-\rangle = |1,0\rangle \otimes |1,-1\rangle$$
$$|\pi^0\pi^-\rangle = \tfrac{1}{\sqrt{2}}\,|2,-1\rangle + \tfrac{1}{\sqrt{2}}\,|1,-1\rangle$$

$$|\pi^+\rangle \otimes |\pi^-\rangle = |1,+1\rangle \otimes |1,-1\rangle \qquad (0,\ I_3 = 0,\ I = 2, 1, 0$$
$$|\pi^+\pi^-\rangle = \tfrac{1}{\sqrt{6}}\,|2,0\rangle + \tfrac{1}{\sqrt{2}}\,|1,0\rangle + \tfrac{1}{\sqrt{3}}\,|0,0\rangle$$

$$|\pi^-\rangle \otimes |\pi^+\rangle = |1,-1\rangle \otimes |1,+1\rangle$$
$$|\pi^-\pi^+\rangle = \tfrac{1}{\sqrt{6}}\,|2,0\rangle - \tfrac{1}{\sqrt{2}}\,|1,0\rangle + \tfrac{1}{\sqrt{3}}\,|0,0\rangle$$

π^0 $|1,0\rangle$ $\otimes$ $|2, 1, 0|\ 0\rangle$

↳ We can invert this relation leading to the decomposition of isospin states into 2 pions.

$$\cancel{|2,2\rangle = |1,+1\rangle = |\pi^+,\pi^+\rangle}$$
$$\cancel{|2,1\rangle = \tfrac{1}{\sqrt{2}}\,|1,0\rangle + \tfrac{1}{\sqrt{2}}\,|0,1\rangle = \tfrac{1}{\sqrt{2}}\,|}$$

$$|2,2\rangle = |1,1\rangle \otimes |1,1\rangle = |\pi^+\pi^+\rangle$$

$$|2,1\rangle = \tfrac{1}{\sqrt{2}}\,|1,1\rangle \otimes |1,0\rangle + \tfrac{1}{\sqrt{2}}\,|1,0\rangle \otimes |1,1\rangle = \tfrac{1}{\sqrt{2}}\,|\pi^+\pi^0\rangle + \tfrac{1}{\sqrt{2}}\,|\pi^0,\pi^+\rangle$$

$$|2,0\rangle = \tfrac{1}{\sqrt{6}}\,|1,1\rangle \otimes |1,-1\rangle + \sqrt{\tfrac{2}{3}}\,|1,0\rangle \otimes |1,0\rangle + \tfrac{1}{\sqrt{6}}\,|1,-1\rangle \otimes |1,1\rangle$$
$$= \tfrac{1}{\sqrt{6}}\,|\pi^+\pi^-\rangle + \sqrt{\tfrac{2}{3}}\,|\pi^0,\pi^0\rangle + \tfrac{1}{\sqrt{6}}\,|\pi^-\pi^+\rangle$$

$$|1,\pm 1\rangle = \pm \frac{1}{\sqrt{2}} |\pi^{\pm}\pi^{0}\rangle \mp \frac{1}{\sqrt{2}} |\pi^{0}\pi^{\pm}\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}} |\pi^{+}\pi^{-}\rangle - \frac{1}{\sqrt{2}} |\pi^{-}\pi^{+}\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{3}} |1,1\rangle \otimes |1,-1\rangle - \frac{1}{\sqrt{3}} |1,0\rangle \otimes |1,0\rangle + \frac{1}{\sqrt{3}} |1,-1\rangle \otimes |1,1\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{3}} |\pi^{+}\pi^{-}\rangle - \frac{1}{\sqrt{3}} |\pi^{0}\pi^{0}\rangle + \frac{1}{\sqrt{3}} |\pi^{-}\pi^{+}\rangle$$

Since the pions are bosons (spin 0) they obey Bose statistics
⇒ the wave function of the state is even (odd) under exchange of 2 pions if the angular momentum quantum number ℓ is even (odd)

From above we read off the angular momentum is even for the states with $I=0$ or 2 and odd for the state with $I=1$.

→ More formally rotation.

↳ Similar to what we did with 2 pions we decompose the state $|0,0\rangle$ into 3 pions.
We start with the 3rd component of isospin (we learn it is additive) → we have to sum the 3 I_3 to 0:
What can it be?

- $3\ I_3 = 0$ $\quad |\pi^0,\pi^0\pi^0\rangle$
- $I_3 = 1,\ I_3 = -1,\ I_3 = 0$ $\quad |\pi^+\pi^-\pi^0\rangle$

↳ The only the η decay channels allowed by isospin.
N.B.: Of course charge conservation also prohibits all the other channels

Exercise: What about J/ψ? $c\bar{c}$

We construct a 3 pion state by coupling a 2 pion states to a single pion

A single pion has total isospin $I = 1$, a 3 pion state with vanishing total isospin can only be achieved if the 2 pion state has $I = 1$ as well

$$|0,0\rangle_3 = \frac{1}{\sqrt{3}} |1,1\rangle_1 |1,-1\rangle_2 - \frac{1}{\sqrt{3}} |1,0\rangle_1 |1,0\rangle_2 + \frac{1}{\sqrt{3}} |1,-1\rangle_1 |1,1\rangle_2$$

number of pions

From before read off the pion states:

$$|0,0\rangle = \frac{1}{\sqrt{3}} \left(|\pi^+\rangle \otimes \left(-\frac{1}{\sqrt{2}} |\pi^- \pi^0\rangle + \frac{1}{\sqrt{2}} |\pi^0 \pi^-\rangle \right) \right)$$

$$- \frac{1}{\sqrt{3}} \left(|\pi^0\rangle \otimes \left(\frac{1}{\sqrt{2}} \left(|\pi^+ \pi^-\rangle - |\pi^- \pi^+\rangle \right) \right) \right)$$

$$+ \frac{1}{\sqrt{3}} \left(|\pi^-\rangle \otimes \left(\frac{1}{\sqrt{2}} \left(|\pi^+ \pi^0\rangle - |\pi^0 \pi^+\rangle \right) \right) \right)$$

$$= \frac{1}{\sqrt{6}} \Big(|\pi^+ \pi^0 \pi^-\rangle - |\pi^+ \pi^- \pi^0\rangle + |\pi^0 \pi^- \pi^+\rangle - |\pi^0 \pi^+ \pi^-\rangle + |\pi^- \pi^+ \pi^0\rangle - |\pi^- \pi^0 \pi^+\rangle \Big)$$

The state containing $3\pi^0$ does not contribute at all because $2\pi^0$ can't form a $|1,0\rangle$ state.

charged angular momentum

As the 3 pion state is anti-symmetric in the pions (change 2 pions / commute → introduce minus sign) it has according to Bose statistics ⇒ odd angular momentum

In particular The angular momentum must be non-zero! ⇒ we conclude that the decay is not allowed if isospin is conserved.

J

But happen in nature

slide.

$J = S + L$, $S = 0$, orbital angular momentum

$\boxed{\text{PDG}}$ $\eta \to \pi^+ \pi^- \pi^0$ 28% ~~of charged~~ ⇒ Isospin is broken

↳ responsible for difference of mass between neutron and proton!

$\eta \to 3\pi$ is directly proportional to $(m_u - m_d)$

$\Rightarrow$ Forbidden by isospin symmetry. 3 pions can not at the same time couple to vanishing angular momentum and zero isospin

The only operator in the QCD Lagrangian that can produce such a transition is

$$\boxed{\mathcal{L}_{IB} = -\frac{m_u - m_d}{2}(u\bar{u} - d\bar{d})}$$

$$\mathcal{L}_{QCD} = -\not{\partial} m \bar{q} q$$
$$= -m_u \bar{u} u - m_d \bar{d} d$$

add and substract

$$\mathcal{L}_{QCD} = -\bar{q} M q \qquad M = \sum_{a=0}^{8} \lambda_a M_a \qquad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

$$= -\sum_f \bar{q}_f m_f q_f \qquad M_a = \frac{1}{2} \mathrm{Tr}(\lambda_a M)$$

$$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$M_a = \frac{1}{2} \mathrm{Tr}($

$$\lambda_0 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

4.17?

$$M = \frac{m_u + m_d + m_s}{\sqrt{6}} \lambda^0 + \frac{m_u - m_d}{2} \lambda^3 + \frac{\hat{m} - m_s}{\sqrt{3}} \lambda^8 \qquad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow$ From the term proportional to $\lambda^3 \Rightarrow$ contribution to the Lagrangian

$$\mathcal{L}_{IB} = -\frac{m_u - m_d}{2} \bar{q} \lambda^3 q$$

$$= -\frac{m_u - m_d}{2} (u\bar{u} - d\bar{d})$$

It is a $\Delta I = 1$ operator

$\Rightarrow$ Decay amplitude ~~can be used~~ proportional to $(m_u - m_d)$

can be used to extract this quantity.

Decay width a measure of the size of Isospin breaking in QCD

. Due to difference in electric charges of the up and down quarks ~~the~~ EM interactions violate also isospin and can contribute to $\eta \to 3\pi$.

$\Rightarrow$ Contribution predicted to be small

- * In chiral limit $m_q \to 0$, $\eta \to 3\pi$ vanishes
  $\sim$ no contributions of $O(e^2)$ (Sutherland)
- * At one loop computed [~~Kubis~~ Ditsche, Kubis and Meissner
  corrections $O(e^2 \cdot m_q)$ $\to$ effectively $O(e^2 m_{u,d})$
  not $O(e^2 m_s)$ $\to$ very small $\sim 1\%$

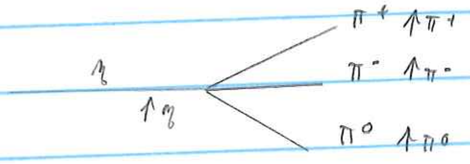
A compléter

$\Rightarrow$ very clean access from $\Gamma(\eta \to 3\pi)$ to $(m_d - m_u)$!

- ChPT at tree level $O(p^2)$ (current algebra)

$$\boxed{A_{\eta\to\pi^+\pi^-\pi^0}(s,t,u) = \frac{B_0(m_u - m_d)}{3\sqrt{3}\,F_\pi^2}\left(1 + \frac{3(s-s_0)}{M_\eta^2 - M_\pi^2}\right)}$$

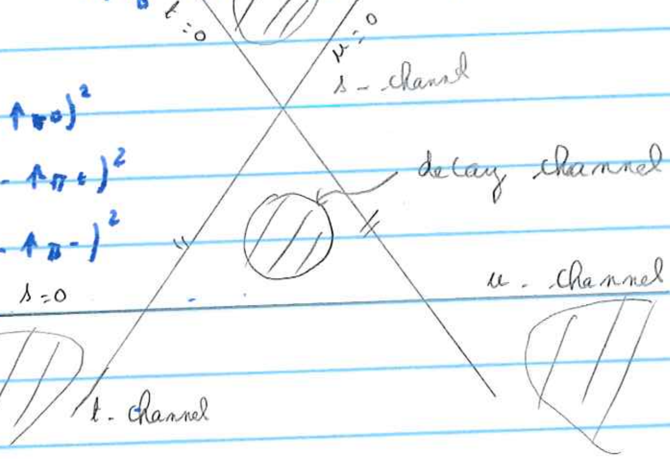
Exercise: show this result



with $A_{\eta\to\pi^+\pi^-\pi^0}$

$$\langle \pi^+\pi^-\pi^0_{out} | \eta \rangle = i(2\pi)^4\,\delta^4(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0})\; A(s,t,u)$$

$$s = (p_{\pi^+} + p_{\pi^-})^2 = (p_\eta - p_{\pi^0})^2$$
$$t = (p_{\pi^0} + p_{\pi^-})^2 = (p_\eta - p_{\pi^+})^2$$
$$u = (p_{\pi^0} + p_{\pi^+})^2 = (p_\eta - p_{\pi^-})^2$$



Now $s+t+u$
$= m_\eta^2 + m_{\pi^0}^2 + 2m_{\pi^+}^2$
$= 3s_0 \to$ Only 2 independant variables.

Digression ...

| For the scattering

$$A_{\eta\to\pi^+\pi^-\pi^0}(s,t,u) = \frac{B_0(m_u - m_d)}{3\sqrt{3}\,F_\pi^2}\left(1 + \frac{3(s-s_0)}{M_\eta^2 - M_\pi^2}\right)$$

Remember: $B_0(m_d - m_u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}(M_K^2 - M_\pi^2)$ at tree level.

$$A_{\eta\to\pi^+\pi^-\pi^0}(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}\,F_\pi^2}\cdot M(s,t,u)$$

$$\boxed{M(s,t,u) = 1 + \frac{3(s-s_0)}{M_\eta^2 - M_\pi^2}}$$

In the center of the Dalitz plot $s=t=u=s_0$

Normalisation $M^{CA}(s=t=u=s_0) = 1$

Remember $\boxed{\frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}}$ ← quantity we want to determine from studying $\eta \to 3\pi$

PDG kinematics (46.21)

One gets $\Gamma = \frac{1}{256\pi^3 m_\eta^3} \int ds\, dt\, |A(s,t,u)|^2$

soit $$\boxed{\Gamma = \frac{1}{Q^4} \frac{m_K^4 (m_K^2 - m_\pi^2)^2}{6912\, \pi^3 m_\eta^3 m_\pi^4 F_\pi^4} \underbrace{\int ds\, dt\, |M(s,t,u)|^2}_{\text{computed with the best accuracy.}}}$$

experimentally measured

↳ determine Q with the best accuracy!

Moderate precision in measurement of Γ → good accuracy on Q since enter as Q^4!

$6^1 =$

$6^2 =$

$6^3 =$

<u>Description of $\eta \to 3\pi$ amplitude</u> $\eta \to 3\pi$ analog to $\pi\pi$

$$M^{ijk}_{\eta\to 3\pi} = M^{ijk,\ell}_{\eta\to 3\pi} \propto \langle \pi^i \pi^j \pi^k | \mathcal{L}_{IB} | \eta \rangle$$
$$\propto \frac{m_u - m_d}{2} \langle \pi^i \pi^j \pi^\ell | \underbrace{\bar{q}\, \tau^\ell q}_{\bar{u}u - \bar{d}d} | \eta \rangle$$

$\boxed{\pi^3 = \pi^0}$

The operator $\bar{q}\, \tau^{\ell=3} q$ transforms under isospin in the same way as pion π^0

The function $M^{ijk}_{\eta\to3\pi}$ transforms exactly like $\pi\pi$ amplitude → share the same isospin structure.

$$\boxed{M^{ijk}_{\eta\to 3\pi} = M^{ijk,3}_{\eta\to 3\pi} = M(s,t,u)\,\delta^{ij}\delta^{k3} + M(t,u,s)\,\delta^{ik}\delta^{j3} + M(u,s,t)\,\delta^{i3}\delta^{jk}}$$

$$M^{+-0}_{\eta\to3\pi}(s,t,u) = \overset{\text{isospin}}{M^{113}_{\eta\to 3\pi}}(s,t,u) = M(s,t,u)$$

$$M^{000}_{\eta\to3\pi}(s,t,u) = M^{333}_{\eta\to3\pi}(s,t,u) = M(s,t,u) + M(t,u,s) + M(u,s,t)$$

Compute

$M^{CA}_{\eta \to 3\pi^0} =$

Exercise : From $M^{CA}_{\eta \to \pi^+\pi^-\pi^0}(s,t,u) = 1 + \frac{3(s-s_0)}{M^2_\eta - M^2_\pi}$

Compute $M^{CA}_{\eta \to 3\pi^0}$.

$$M^{CA}_{\eta \to 3\pi^0}(s,t,u) = 3 + \frac{3(s-s_0)}{M^2_\eta - M^2_\pi} + \frac{3(t-s_0)}{M^2_\eta - M^2_\pi} + \frac{3(u-s_0)}{M^2_\eta - M^2_\pi}$$

$$= 3\left(1 + \frac{(\overbrace{s+t+u}^{3s_0} - 3s_0)}{M^2_\eta - M^2_\pi}\right)$$

$$= 3$$

At LO $M^{CA}_{\eta \to 3\pi^0}(s,t,u)$ is constant !

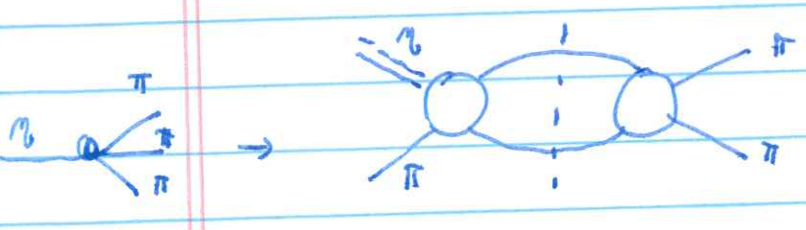
- We want to compute $M_{\eta \to 3\pi}$ beyond LO.

$\eta \to 3\pi$ has a very similar structure as $\pi\pi$ scattering.

At LO linear in $s/t/u$

$\to$ S and P waves only

Consequence for imaginary parts :



cut contribution of D and higher waves requires to be $O(p^4)$ $\to$ complete diagram $O(p^8) \hat{=} 3$ loop order

$\to$ Up to and including 2 loops $\pi\pi$ scattering / $\eta \to 3\pi$ only have discontinuities in S and P waves

- "Reconstruction theorem" : Both $\pi\pi \to \pi\pi$ / $\eta \to 3\pi$ can be decomposed in terms of single variable functions with a right hand cut as :

$$M(s,t,u) = M_0(s) + (s-u)\, M_1(t) + (s-t)\, M_1(u)$$
$$+ M_2(t) + M_2(u) - \frac{2}{3} M_2(s)$$

indices 0, 1, 2 : $\pi\pi$ isospin , $I = 0, 2$ S wave
$I = 1$ P wave.

$s - u = 4 p_t^2 \cos\theta_t \to$ P wave characteristic!

→ see Aitchison's talk

- Use of dispersion relations to describe M_0, M_1 and M_2

why ?

Large final state interactions in $\eta \to 3\pi$ prediction for the width

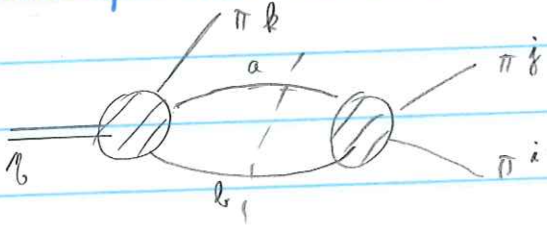
$$\frac{\Gamma^{1\text{-loop}} = 300\ \text{eV}}{\Gamma^{\text{tree}} = 66\ \text{eV}} \sim 2.5 \,!$$

4.5 (85)

- Dispersion relations : unitarity + analiticity + crossing

Start with unitarity : Optical theorem

. Intuitive picture



- Rescattering in "all 3 channels" (s, t, u)
- Everything is projected onto the right (S, P partial waves)

$$\text{disc}\, M^{ijk}_{\eta\to3\pi}(s,t,u) \propto \int d_{LIPS} \Big\{ M^{abk}_{\eta\to3\pi}(s, t'_s, u'_s) \cdot T^{ab,ij}_{\pi\pi}(s, \theta_s)^*$$
$$+ M^{ajb}_{\eta\to3\pi}(s'_t, t, u'_t)\, T^{ab,ik}_{\pi\pi}(t, \theta_t)^*$$
$$+ M^{iab}_{\eta\to3\pi}(s'_u, t'_u, u)\, T^{ab,jk}_{\pi\pi}(u, \theta_u)^* \Big\}$$

we can decompose ππ state in isospin 0, 1 and 2

so $A^{kl,ij}_{\pi\pi\to\pi\pi} = P_0^{kl,ij}\, F_0(s,t,u) + P_1^{kl,ij}\, F_1(s,t,u) + P_2^{kl,ij}\, F_2(s,t,u)$

with $\boxed{|\pi^i\pi^j\rangle_I = \underset{\substack{\uparrow\\ \text{projector operator.}}}{P_I^{ij,kl}}\, |\pi^k\pi^l\rangle}$

Unitarity:

$$\Im m\, \underbrace{A^{kl,ij}_{\pi\pi\to\pi\pi}}_{\text{insert}} = \frac{(2\pi)^4}{4}\sum_{m,n=1}^{3}\int \frac{d^3q_1\, d^3q_2}{4 q_1^0\, q_2^0\, (2\pi)^6}\, \delta^4(q_1+q_2-p_1-p_2) \cdot A^{kl,mn}_{\pi\pi\to\pi\pi}\, A^{*\,mn,ij}_{\pi\pi\to\pi\pi}$$

LHS

$$\Im m\, A^{kl,ij}_{\pi\pi\to\pi\pi} = P_0^{kl,ij}\, \Im m\, F_0 + P_1^{kl,ij}\, \Im m\, F_1 + P_2^{kl,ij}\, \Im m\, F_2$$

RHS:

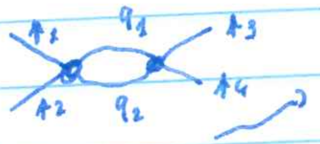
$$\frac{1}{64\pi^3}\int \frac{d^3q_1\, d^3q_2}{q_1^0\, q_2^0}\, \delta^4(q_1+q_2-p_1-p_2)\, \left(P_0^{kl,ij}\, F_0 F_0^* + P_1^{kl,ij}\, F_1 F_1^* + P_2^{kl,ij}\, F_2 F_2^*\right)$$

we have used $P_I^{mn,kl}\, P_J^{kl,ij} = \delta_{IJ}\, P_I^{mn,ij}$

$P_I^{kl,ij} = P_I^{ij,kl}$

Equating LHS and RHS

$$\Im m\, F_I(p_1,p_2,p_3,p_4) = \frac{1}{64\pi^2}\int \frac{d^3q_1\, d^3q_2}{q_1^0\, q_2^0}\, \delta^4(q_1+q_2-p_1-p_2) \cdot F_I(q_1,q_2,p_3,p_4)\, F_I^*(p_1,p_2,q_1,q_2)$$



$\theta_s = (\widehat{\vec{p}_1, \vec{p}_3})$

$$\Im m\, F_I(s,\theta_s) = \frac{1}{128\pi^2}\int d\Omega\, \sqrt{\frac{s-4m_\pi^2}{s}}\, F_I(s,\theta_1)\, F_I^*(s,\theta_2)$$

We expand F_I into partial waves

$$\boxed{F_I(s,\theta) = \sum_{l=0}^{+\infty} (2l+1)\, \underbrace{P_l(\cos\theta)}_{\text{legendre polynomials}}\, f_l^I(s)} \qquad f_l^I(s) = \int_{-1}^{1}$$

legendre polynomials are normalized as

$$\boxed{\int_{-1}^{1} dx\, P_\ell(x)\, P_{\ell'}(x) = \frac{2}{2\ell+1}\, \delta_{\ell\ell'}}$$

They satisfy $\int d\Omega\, P_\ell(\cos\theta_1)\, P_\ell(\cos\theta_2') = \frac{4\pi}{2\ell+1}\, \delta_{\ell\ell'}\, P_\ell(\cos\theta)$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

so $\sum_{\ell=0}^{+\infty} (2\ell+1)\, P_\ell(\cos\theta)\, \mathrm{Im}\, f_\ell^I(s) = \frac{1}{128\pi^2} \int d\Omega \sqrt{\frac{s - 4m_\pi^2}{s}}$

$\sum_{m=0}^{+\infty} (2m+1)\, P_m(\cos\theta_1) \sum_{n=0}^{+\infty} (2n+1)\, P_n(\cos\theta_2)$

$\frac{1}{128\pi^2}\, \frac{4\pi}{2m+1} \sqrt{\frac{s-4m_\pi^2}{s}}\, (2m+1)(2m+1)$

→ They are linearly independant

$\mathrm{Im}\, f_\ell^I(s) = \sqrt{\frac{s - 4m_\pi^2}{s}}\, \frac{1}{32\pi}\, f_\ell^I(s)\, f_\ell^{I*}(s)$

$|f_\ell^I(s)|\, e^{i\delta_\ell^I(s)}$ $|f_\ell^I(s)|\, e^{-i\delta_\ell^I(s)}$

$|\cancel{f_\ell^I(s)}|\, \sin\delta_\ell^I(s) = \sqrt{\frac{s - 4m_\pi^2}{s}}\, \frac{1}{32\pi}$

$|f_\ell^I(s)| = 32\pi \sqrt{\frac{s}{s - 4m_\pi^2}}\, \sin\delta_\ell^I(s)$

$$\Rightarrow \boxed{f_\ell^I(s) = \frac{32\pi}{\sqrt{1 - \frac{4m_\pi^2}{s}}}\, \sin\delta_\ell^I(s)\, e^{i\delta_\ell^I(s)}}$$

$F_I(s, \cos\theta_s) = \sum_{\ell=0}^{+\infty} (2\ell+1)\, P_\ell(\cos\theta)\, f_\ell^I(s)$

Bose statistics

ℓ and I both odd or even

truncate the sum to S and P waves
$\ell = 0$ $\ell = 1$

$F_{0,2}(s, \cos\theta_s) = \sum_{\ell=0}^{1} (2\ell+1)\, P_\ell(\cos\theta)\, f_\ell^{0,2}(s)$

$\ell = 1$ not allowed.

$= P_0(\cos\theta)\, f_0^{0,2}(s)$ → $\pi\pi$ $I=0$, $\ell=0$ phase shift

$\pi\pi$ $I=2$ $\ell=0$ neglect "

$\left| T^{I=0,2} = \frac{32\pi}{\sqrt{1 - \frac{4m_\pi^2}{s}}}\, e^{i\delta_{0,2}(s)} + \text{D waves} \ldots \right.$ $\ell = 2$

$$F_1(s, \cos\theta_s) = \sum_{l=0}^{1} (2l+1) P_l(\cos\theta) f_l^1(s)$$

$l=0$ not allowed

$$\boxed{T^{I=1} = 3 \cos\theta \frac{\sin\delta_1(s)\, e^{i\delta_1(s)}}{\sqrt{1-\frac{4m_\pi^2}{s}}}} + F \text{ waves } l=3$$

$\pi\pi$ $I=1$, $l=1$ phase shift

~~Now $M^{+-0}(s,t,u) = M^{113}$~~ → Only with $\pi\pi$ intermediate channel.

On a $\text{disc}\, M^{ijk}_{\gamma\to3\pi}(s,t,u) \propto \int d_{LIPS} \{ T^{ab,ij}_{\pi\pi}(s,\theta_s)^* M^{abk}_{\gamma\to3\pi}(s,t'_s,u'_s)$
$+ T^{ab,ik}_{\pi\pi}(t,\theta_t)^* M^{ajb}_{\gamma\to3\pi}(s'_t, t, u'_t)$
$+ T^{ab,jk}_{\pi\pi}(u,\theta_u)^* M^{iab}_{\gamma\to3\pi}(s'_u, t'_u, u) \}$

$$\theta_s = (\widehat{\vec{p}_a, \vec{p}_3})$$

$$\begin{matrix} t \\ u \end{matrix} = \frac{1}{2}(3s_0 - s \pm \kappa(s)\cos\theta_s)$$

$$\kappa(s) = 4|p_{\pi\pi}||p_{\pi\kappa}| = \frac{\lambda^{1/2}(s, m^2_{\pi\pm}, m^2_{\pi\pm})\lambda^{1/2}(s, m^2_\gamma, m^2_{\pi 0})}{s}.$$

$$= \sqrt{1-\frac{4m_\pi^2}{s}}\, \lambda^{1/2}(s, m_\gamma^2, m_\pi^2)$$

$\cos\theta_s = \frac{t-u}{\kappa(s)}$

→ we want to describe the charged channel:

* $M^{+-0}(s,t,u) = M^{113}(s,t,u) = M(s,t,u)$

$\text{disc}\, M(s,t,u) = \frac{1}{128\pi^2} \sum_{a,b} \int d\Omega \{ \sqrt{1-\frac{4m_\pi^2}{s}}\, T^{ab,11*}_{\pi\pi}(s,\theta_s)$

$\cdot M^{ab3}(s, t'_s, u'_s) + \sqrt{\frac{t-4m_\pi^2}{t}}\, T^{ab,13*}_{\pi\pi}(t,\theta_t) M^{a1b}(s'_t, t, \ldots$

insert decomposition

$+ \sqrt{\frac{u-4m_\pi^2}{u}}\, T^{ab,13*}_{\pi\pi}(u,\theta_u) M^{1ab}(s'_u, t'_u, u) \}$

$d\Omega = d\phi\, d\cos\theta$

→ we decompose the $T_{\pi\pi}$ in isospin.

with $P_0^{kl,ij} = \frac{1}{3}\delta^{kl}\delta^{ij}$

$P_1^{kl,ij} = \frac{1}{2}(\delta^{ki}\delta^{lj} - \delta^{li}\delta^{kj})$

$P_2^{kl,ij} = \frac{1}{2}(\delta^{ki}\delta^{lj} + \delta^{li}\delta^{kj}) - \frac{1}{3}\delta^{kl}\delta^{ij}$

$$\text{disc}\, M(s,t,u) = \frac{1}{4\pi}\int d\Omega \left\{ \frac{1}{3}\sin\delta_0(s)\, e^{-i\delta_0(s)} \left(3M(s,t'_s,u'_s) + M(t'_s,u'_s,s) + M(u'_s,s,t'_s)\right) \right.$$

$$- \frac{1}{3}\sin\delta_2(s)\, e^{-i\delta_2(s)}\left(M(t'_s,u'_s,s) + M(u'_s,s,t'_s)\right)$$

$$+ \frac{3}{2}\sin\delta_1(t)\, e^{-i\delta_1(t)}\cos\theta_t \left(M(s'_t,t,u'_t) - M(u'_t,s'_t,t)\right)$$

$$+ \frac{1}{2}\sin\delta_2(t)\, e^{-i\delta_2(t)}\left(M(s'_t,t,u'_t) + M(u'_t,s'_t,t)\right)$$

$$+ \frac{3}{2}\sin\delta_1(u)\, e^{-i\delta_1(u)}\cos\theta_u\left(M(s'_u,t'_u,u) - M(t'_u,u,s'_u)\right)$$

$$\left. + \frac{1}{2}\sin\delta_2(u)\, e^{-i\delta_2(u)}\left(M(s'_u,t'_u,u) + M(t'_u,u,s'_u)\right)\right\}$$

~~disc~~ $\int d\Omega = \int d\phi\, d\cos\theta$

$$\int d\Omega\, z^n M_I(t'_s) = (-1)^n \int d\Omega\, z^n M_I(u'_s)$$

$$\rightarrow \boxed{M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)}$$

$$\text{disc}\, M_I(s) = \text{disc}\, f_\ell^I(s)$$

Watson theorem $\Rightarrow$ $\text{disc.}\, f_\ell^I(s) \propto \underbrace{t_\ell^{I*}(s)}_{\text{partial wave of scattering matrix}} f_\ell^I(s)$

$$\boxed{f_\ell^I(s) = \underbrace{M_I(s)}_{\text{right hand cut}} + \underbrace{\hat{M}_I(s)}_{\text{left hand cut}}}$$

$\hat{M}_I$ real on the right hand cut

~~Determination of $\hat{M}_I(s)$~~

$$\text{disc}\, M_I(s) = \text{disc}\,[f_\ell^I(s)]$$

$$= 2i\,\theta(s-4M_\pi^2)\, \underset{\left[M_I(s)+\hat{M}_I(s)\right]}{\underset{\parallel}{f_\ell^I(s)}}\; \underset{\sin\delta_\ell^I(s)\, e^{-i\delta_\ell^I(s)}}{\underset{\parallel}{t_\ell^{I*}(s)}}$$

δ_ℓ^I ↓ $\pi\pi$ scattering ph[ase] in elastic.

$$T^{ab,11}_{\pi\pi} = A^{ab,11}_{\pi\pi\to\pi\pi} = \underbrace{P_0^{ab,11}}_{\frac{1}{3}\delta^{ab}} F_0(s,t,u) + \underbrace{P_1^{ab,11}}_{\frac{1}{2}[\delta^{a1}\delta^{b1} - \dots} F_1(s,t,u)$$

$$\overset{\shortparallel}{0}$$

and also $M_{\eta\to3\pi}(s,t,u)$

$$M^{ijk}_{\eta\to3\pi}(s,t,u) = M(s,t,u)\,\delta^{ij}\delta^{k3} + M(t,u,s)\,\delta^{ik}\delta^{j3} + M(u,s,t)\,\delta^{i3}\delta^{jk}$$

14f

→ disc $M_I(s) = 2i\,\Theta(s-4M_\pi^2)\,\{M_I(s) + \hat{M}_I(s)\}\,\sin\delta_I(s)\,e^{i\delta_I^\ell(s)}$

almost like the FF equation: disc $F(s) = 2i\,\Theta(s-4m_\pi^2)\,F(s)\sin\delta(s)\,e^{i\delta(s)}$

except for $\hat{M}_I(s)$.

→ Homogeneitäties see 14 g

We know how to solve $\mathrm{Im}\, M(s) = \Theta(s-4M_\pi^2)\, M(s)\sin\delta(s)\,e^{i\delta(s)}$

Exercise → solution $M(s) = P(s)\,\Omega(s)$

→ How to solve in the general case? Consider discontinuity of $\frac{M(s)}{\Omega(s)}$, homogeneous $= 0$ solution ⇒ polynomial.

<u>Exercise</u>: find the discontinuity of $\frac{M(s)}{\Omega(s)}$

$$\Omega(s) = e^{\frac{s}{\pi}\int_{4m_\pi^2}^{+\infty} \frac{\delta(s')}{s'(s'-s)}ds'} \qquad \Omega(s+i\varepsilon) = e^{i\delta_I^\ell(s) + P\int_{4m_\pi^2}^{\infty}\frac{\delta(s')}{s'(s'-\dots}}$$

$$\Omega(s+i\varepsilon) = e^{\frac{s}{\pi}\int_{4m_\pi^2}^{+\infty}\frac{\delta(s')}{s'(s'-s-i\varepsilon)}ds'}$$

Sokhotsky - Plemelj formula

$$\lim_{\varepsilon\to0}\frac{1}{x\mp i\varepsilon} = \pm i\pi\,\delta(x) + \mathrm{P.P}\,\frac{1}{x}$$

$$\frac{1}{x(x-s)} = -\frac{1}{xs} + \frac{1}{s(x-s)}$$

$$= \left[-\frac{1}{s}\ln x + \frac{1}{s}\ln(x-s)\right]_{4\dots}$$

$$= -\frac{1}{s}\ln\left[1-\frac{s}{x}\right]_{4m_\pi^2}^{\infty}$$

Determination of $\hat{M}_I(s)$:

Substract $M_I(s)$ from partial wave projection of $M(s,t,u)$

$M(s,t,u) = M_0(s) + (s-u)\, M_1(t) + \ldots$

$\hat{M}_I(s)$ singularities in the t and u, depend of other $M_I(s)$

Angular averages of the other functions $\Rightarrow$ coupled equations.

$$\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle + 2(s-s_0)\langle M_1\rangle + \frac{20}{9}\langle M_2\rangle + \frac{2}{3}k(s)\langle z M_1\rangle$$

$\int d\Omega\, M(t(s,z)) = \int_{-1}^{1} dz\, z\, M_1^{*}(s,z)$

where $\langle z^n M_I\rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z))$

$z = \cos\theta$ scattering angle

Non trivial angular averages $\Rightarrow$ need to deform the integration path to avoid crossing cuts

$\rightarrow$ Physics behind ~~the~~ inhomogeities p 15

See slides