Principles of the S-matrix

- Unitarity
- Structure of the T-matrix for complex energies
- Analyticity
- Crossing symmetry
- Dispersive representations

[Gribov]: V.N. Gribov, *Strong interactions of Hadrons at High Energies*, Cambridge University Press, 2008, ISBN 978-0-521-85609-6

[Peskin-Schroeder]: M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum field Theory*, Perseus Books, 1995, ISBN 0-201-50397-2

Elementary particles of the Standard Model

Strong interaction

Particles like the electron (fermions, spin 1/2)

Leptons		Quarks (each in 3 "colors")	
e 0.511 MeV	ν_e < 0.000003	d 7	u 3
μ 106	ν_μ < 0.2	s 120	c 1200
τ 1777	ν_τ < 20	b 4300	t 175,000
-1	0	$-1/3$	$2/3$ ← charge

Particles like the photon (bosons, spin 1)

γ 0	photon	"electromagnetism"
g 0	gluon (8 "colors")	"strong interaction"
$W^{\pm}$ 80,420	Z^0 91,188	"weak interaction"

(Gravity is negligible.)

Questions

- Q: How many quarks or gluons have ever been directly observed?

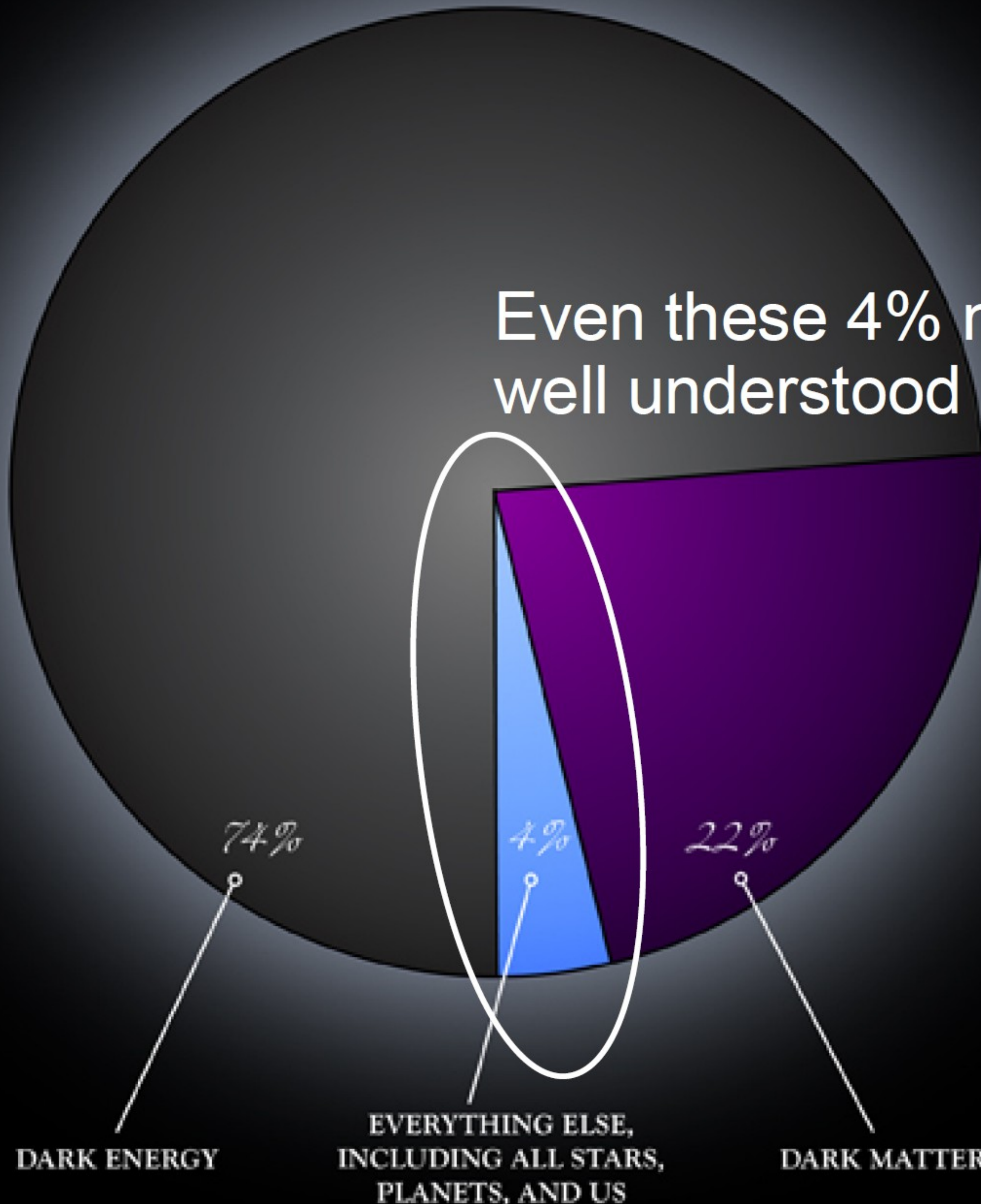
  A: 0 (zero)

- Q: The mass of a down quark is 5 MeV and that of an up quark is 2.3 MeV Then, the mass of the proton (uud) should be $m_P \sim$ 10 MeV, right?

  A: $m_P = 938.272$ MeV

- It is obviously a long way from our "periodic table" of quarks and gluons to matter and its properties as we know them.

Even these 4% not well understood



idea: S. Beane

Quark-gluon interaction: QCD

Remember from Mechanics:

L=T-V

describes your physical system [T: Kinetic energy, V: potential energy]

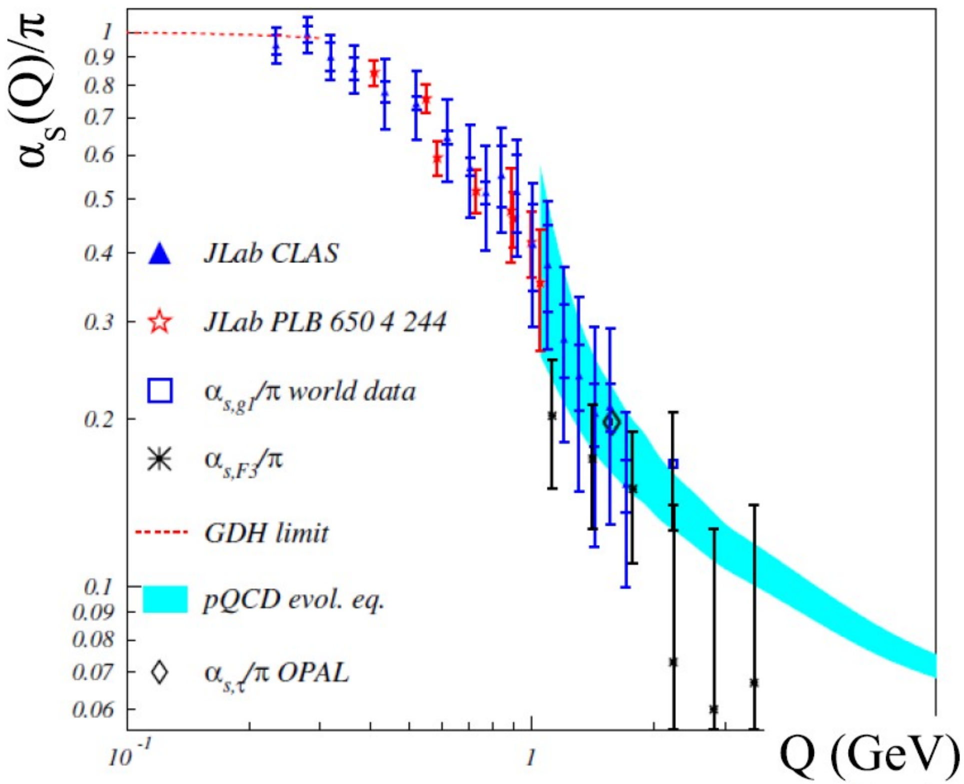
$$\mathcal{L}_{\mathrm{QCD}} = \sum_f \bar{\psi}_f \left[D_\mu \gamma_\mu + m_f \right] \psi_f + \frac{1}{4} \sum_a G^a_{\mu\nu} G^{a\mu\nu}$$

e.g.: $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + \mathbf{g} f^{abc} G^b_\mu G^c_\nu$

One fundamental coupling strength $\boxed{\alpha_S} = \frac{g^2}{4\pi} \ll 1.$

For small α_S, one can solve QCD in a controlled way (perturbation theory).

The running coupling

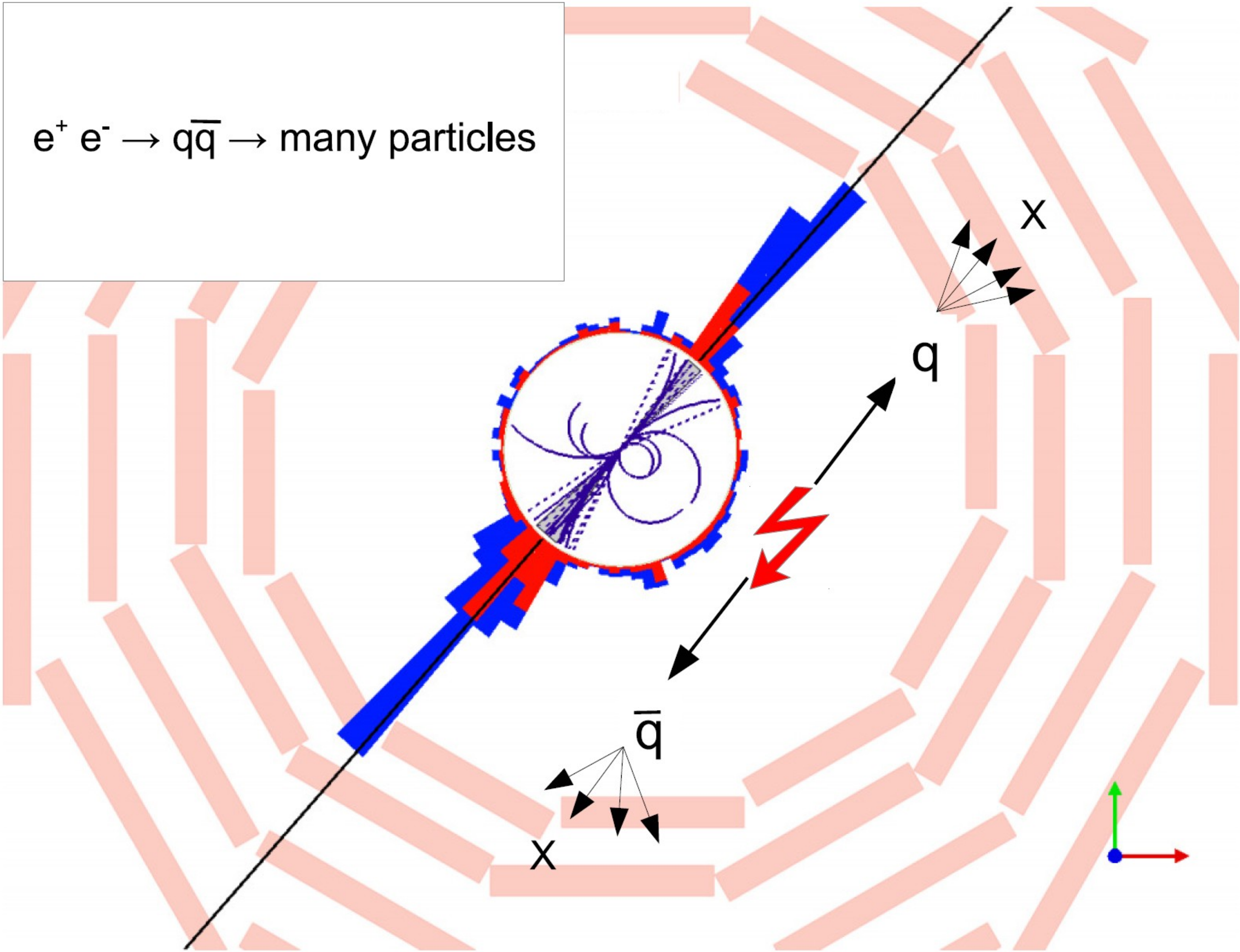


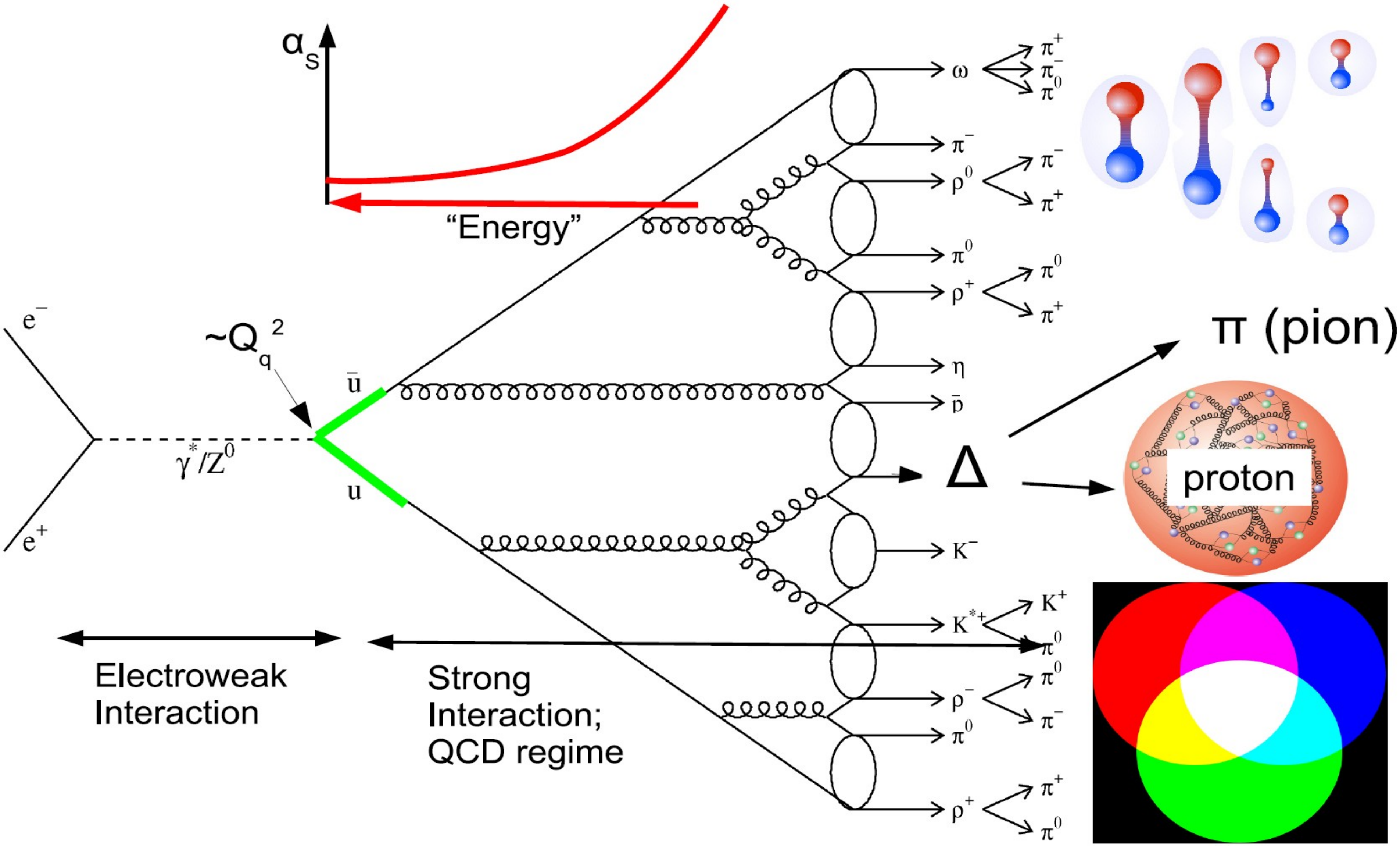
⟵ Energy decreases

Deur, Burkert, Chen, Korsch, PLB665 (2008)

No easy solution of QCD at lower energies!

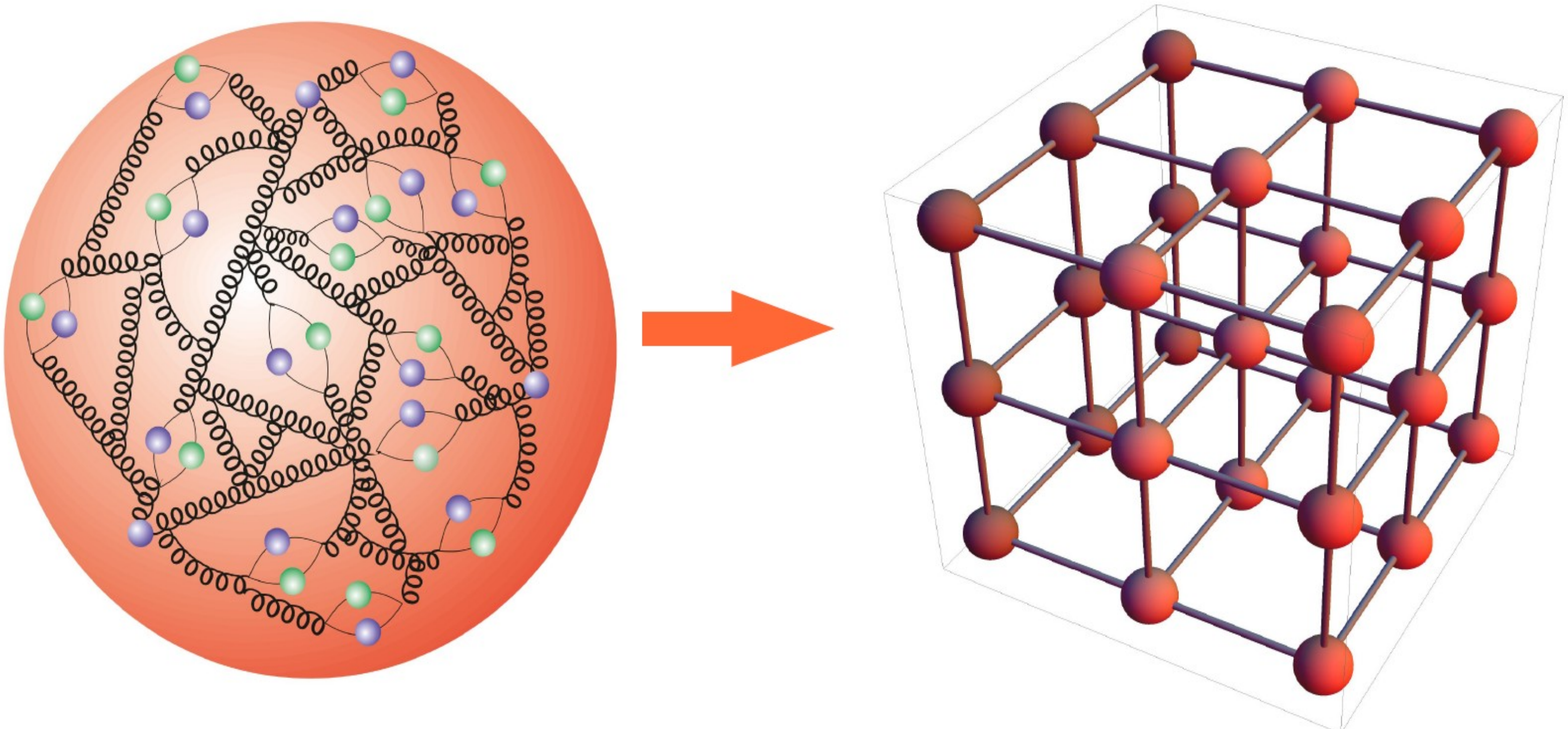
$e^+ e^- \rightarrow q\bar{q} \rightarrow$ many particles





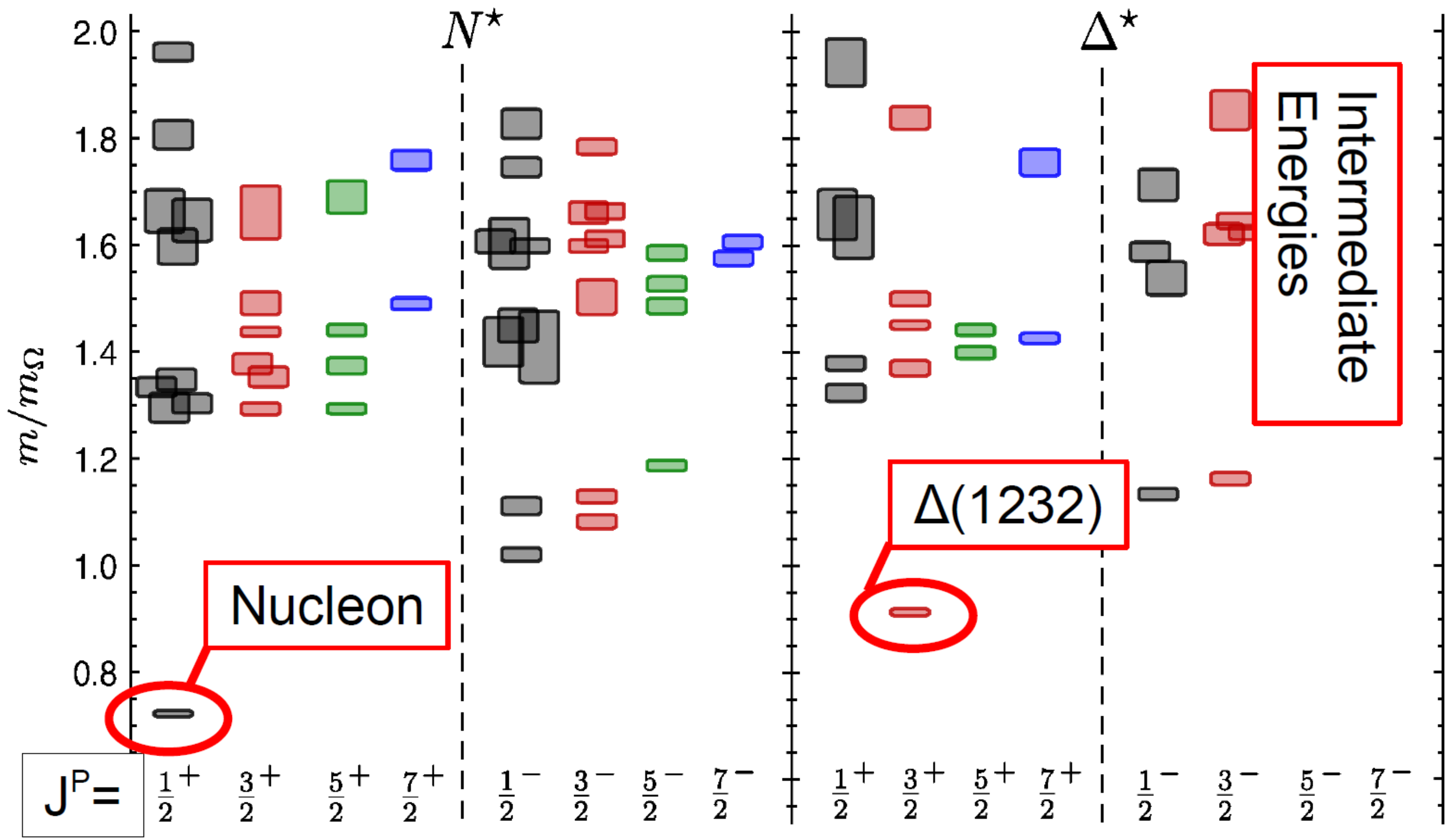
Only colorless final states ↔ confinement

Lattice QCD for hadrons



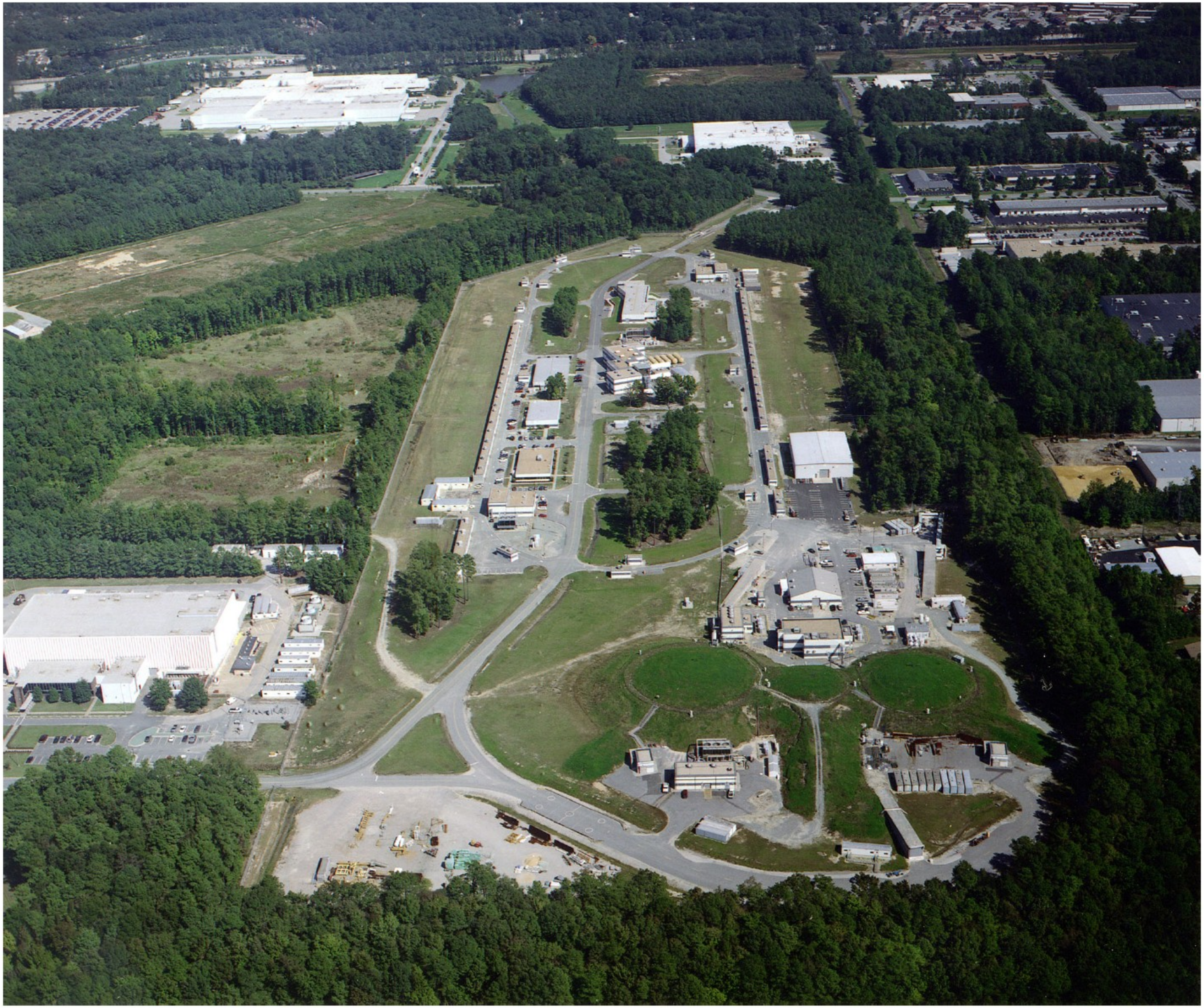
- Simulate the complexity of QCD at low energies with the help of supercomputers
- Ab-initio approach: QCD $\rightarrow$ hadron masses
- Discretization in space and time, in a finite volume, to make the problem numerically treatable

- Many resonances predicted in lattice calculations [Edwards *et al.*, Phys.Rev. D84 (2011)]:



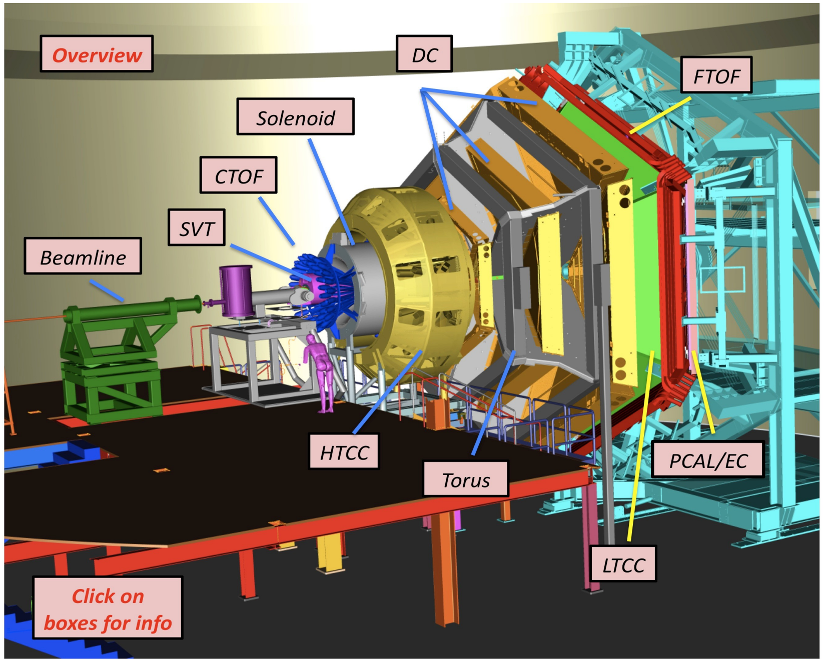
$m_\pi = 396 MeV$ (!)

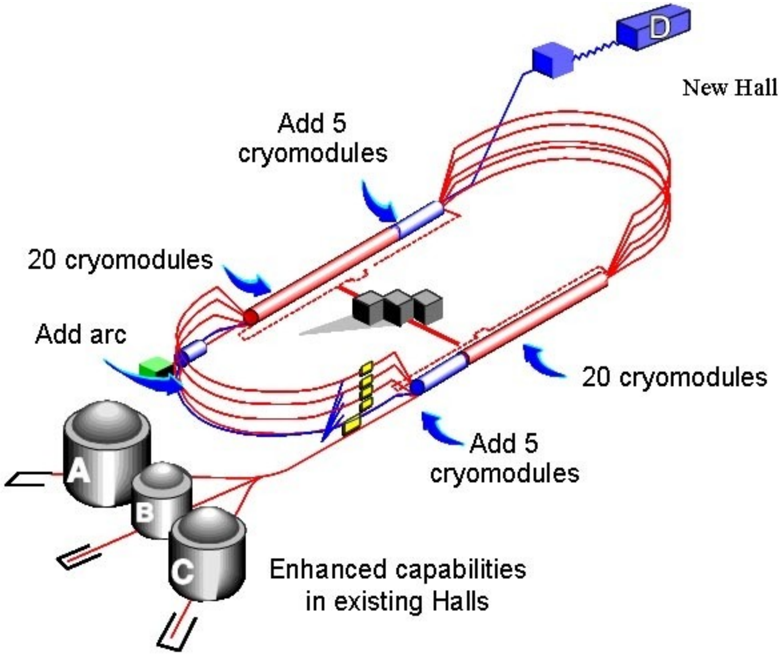
- Search for these states in dedicated experimental programs

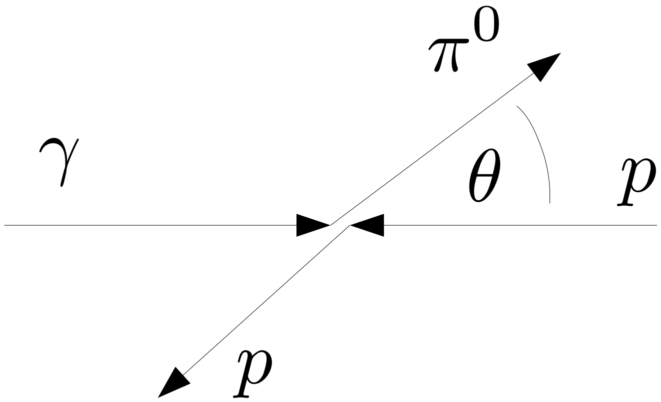


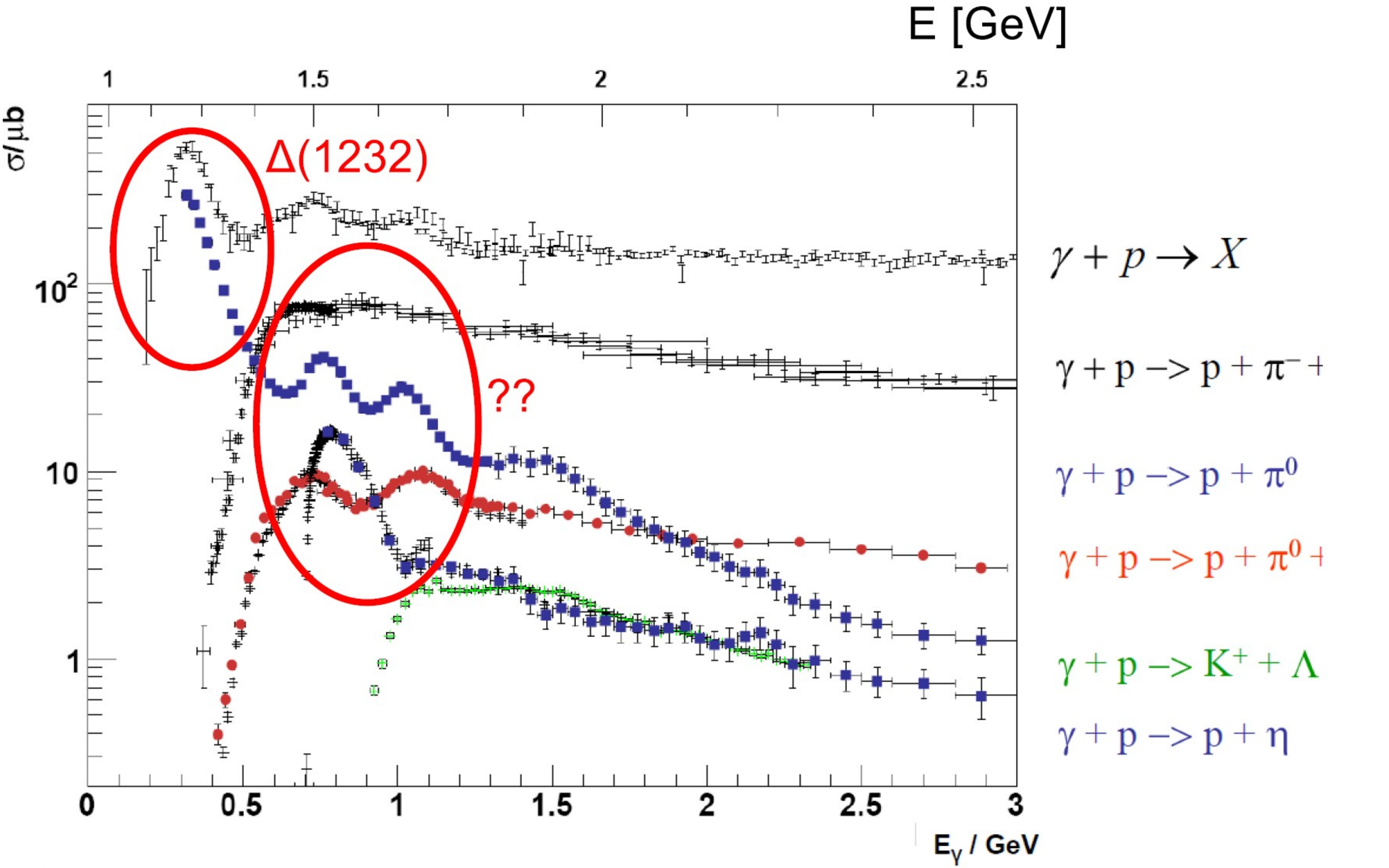
Examples

<u>Fit of Photoproduction Data</u>
(from **Jlab**, VA; MAMI, GE; ELSA, GE)









[data: JLab, ELSA, MAMI]

Photoproduction

Reaction type

Observable:
Differential cross section
$\frac{d\sigma}{d\Omega}\,[\mu b/sr]$

Data from
different
experiments

Models from different
analysis groups
(GW-based and others)

Excluded regions

large variations
~ 2 orders of magnitude

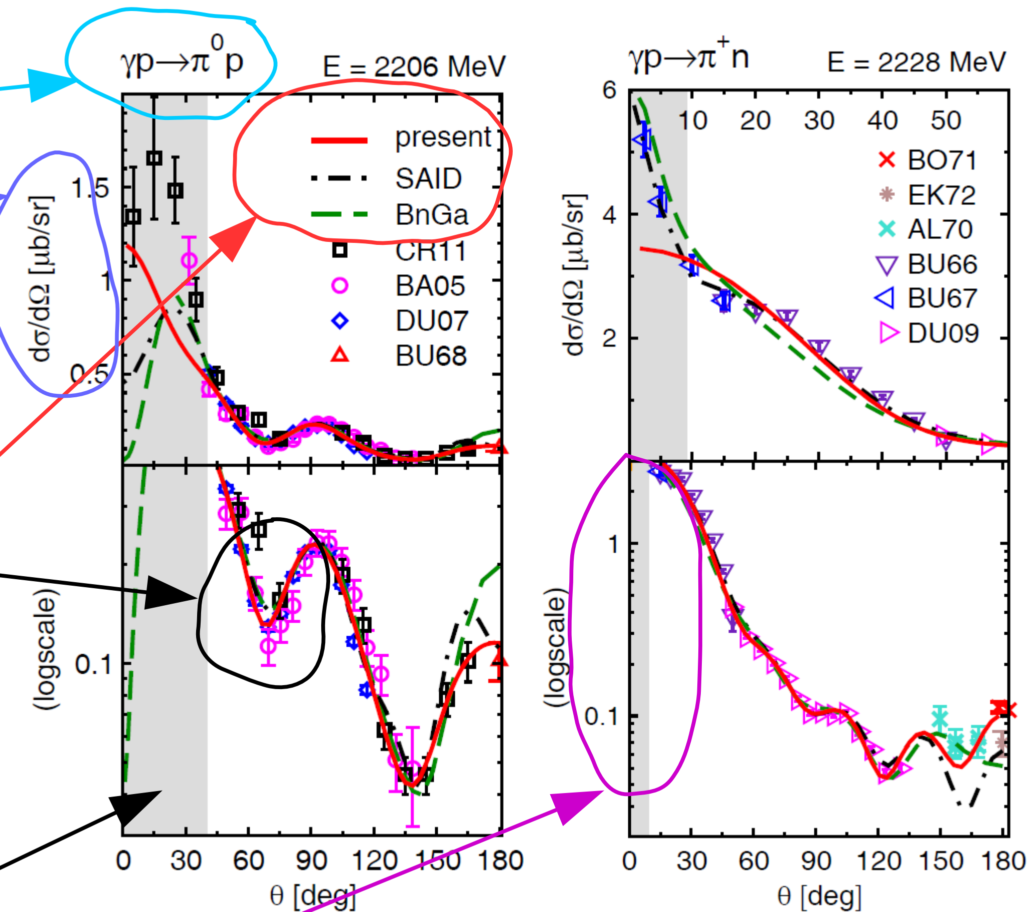
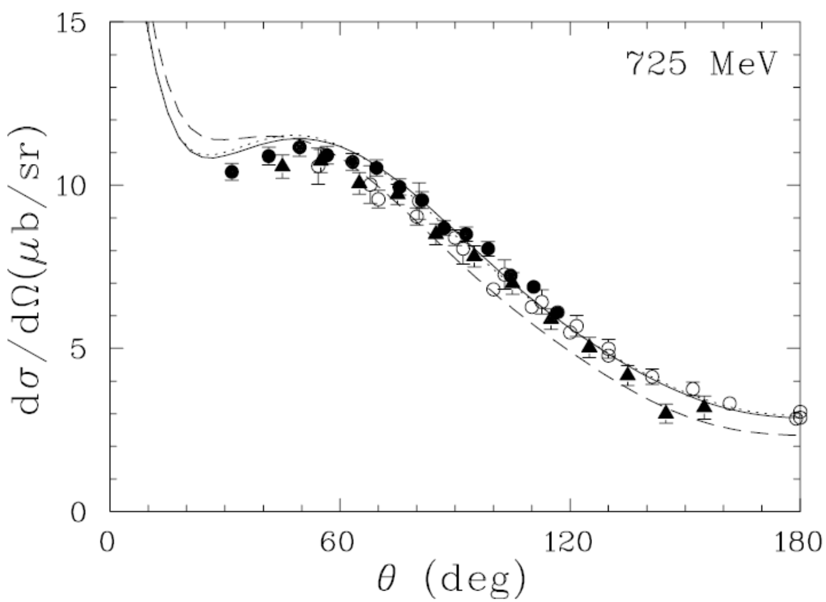


Fig. 2. High-energy behavior in the reaction $\gamma p \to \pi^0 p$ (left) and $\gamma p \to \pi^+ n$ (right). Solid (red) line: fit 2; dash-dotted (black) line: GWU/SAID CM12 [3]; dashed (green) line: Bonn-Gatchina [119]. Data $\pi^0 p$: CR11 [120], BA05 [126], DU07 [127], BU68 [128]. Data $\pi^+ n$: BO71 [129], EK72 [130], AL70 [131], BU66 [132], BU67 [133], DU09 [134]. The regions excluded in our fit are shown as shaded areas.

Partial wave analysis from many observables: $\gamma N \to \pi N$

Differential cross section $\gamma p \to n\pi^+$



[CLAS measurements, PRC 79 (2009); Solid (dashed) lines: SAID (MAID) analysis; filled: CLAS, triangles: MAMI

- Photon: Spin 1
  Nucleon: Spin 1/2
- Single, double, triple polarization observables
- Order principle in the chaos: Conserved quantum numbers, e.g.,
  J^P: Total angular momentum^Parity

Partial wave analysis

Decompose experimental data with respect to conserved quantum numbers.
Resonances have a certain, conserved J^P.

The S-matrix

- Transition from some initial state *a* to some final state *b:*

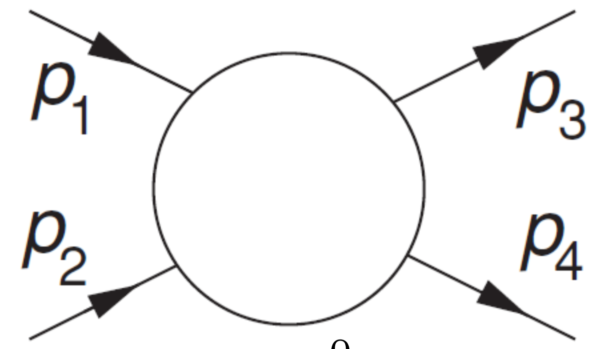
$$S = \mathrm{I} + i\,T; \quad S_{ab} = \delta_{ab} + i\,T_{ab}$$

- With incoming particle *i* and outgoing particles *j*:

$$T_{ab} = (2\pi)^4\,\delta^4\left(\sum_{i\in a} p_i - \sum_{j\in b} k_j\right)\prod_{i\in a}\frac{1}{\sqrt{2p_{0i}}}\prod_{j\in b}\frac{1}{\sqrt{2k_{0j}}}\cdot\mathcal{M}_{ab}$$

  - energy momentum conservation
  - wave function renormalization: factors of $1/\sqrt{2p_0}$
  - "*T*"-matrix and Lorentz invariant amplitude $\mathcal{M}_{ab}$
- Reaction probability: square *T*!

2--> 2 Scattering and the Mandelstam plane

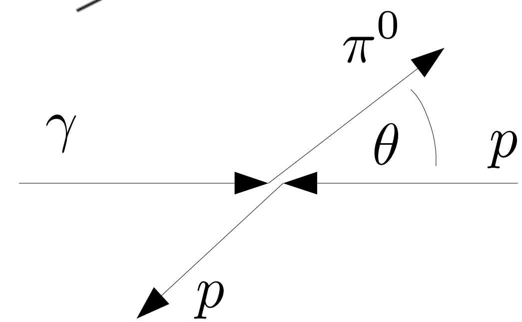


- On-mass-shell external particles:

$$p_i^2 = m_i^2$$

- Two independent kinematical

  variables (e.g., scattering angle and energy):



  Three four-vectors (12 components)

  Four on-mass-shell conditions

  Three rotations and three Lorentz boosts. Altogether: 12-4-3-3=**2**

  Two independent kinematic variables to characterize the invariant amplitude in 2--> 2 scattering

- Similarly: for a 2 → 3 process, 5 independent kinematic variables

Symmetries of the strong interaction

- Electric charge Q
- Baryon charge (baryon number conservation)

  +1 for $p,\ n,\ \Lambda,\ \Sigma,\ \Xi,\ \ldots$

  -1 for anti-particles

  0 for mesons $(\pi,\ K,\ \rho,\ \omega,\ \varphi,\ \ldots)$
- Isotopic spin (isospin) I approximately

$$\frac{m_n - m_p}{m_p} \sim \frac{m_{\pi^0} - m_{\pi^+}}{m_{\pi^+}} \sim \alpha \simeq \frac{1}{137}$$

- Strangeness S: Lambda's and Kaons always produced together:

$$\pi^- + p \rightarrow \Lambda + \bar{K}^0$$

but never observed: $\pi^- + p \rightarrow n + K^0, \quad \text{or} \quad \pi^- + p \rightarrow \Lambda + \pi^0$

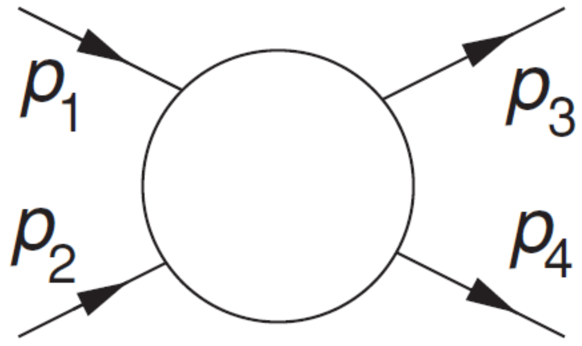
Gell-Mann-Nishijima $$Q = I_3 + \frac{B}{2} + \frac{S}{2}.$$

+ SU(3) symmetry + ...

Mandelstam variables

- Characterize kinematics through Mandelstam variables:

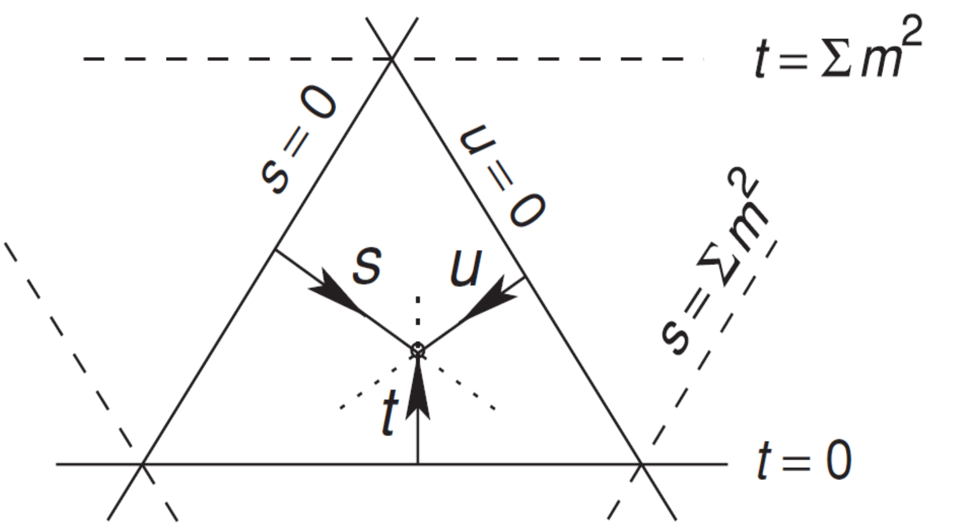
$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \end{aligned}$$



- Using on-mass-shell condition, one gets immediately

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

- Can be visualized in the Mandelstam plane:



Meaning of Mandelstam variables

- Choose center-of-mass (cm) frame: $\mathbf{p}_1 + \mathbf{p}_2 = 0$

$$s = (p_{1\mu} + p_{2\mu})^2 \equiv (p_{10} + p_{20})^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 = (E_{1c} + E_{2c})^2 = E_c^2$$

Square of the energy of total energy of colliding particles

- Express t and u through scattering angle:

$$t = (p_{3\mu} - p_{1\mu})^2 \equiv (E_3 - E_1)^2 - (\mathbf{p}_3 - \mathbf{p}_1)^2$$

$$= (E_3 - E_1)^2 - (\mathrm{p}_3 - \mathrm{p}_1)^2 - 2\mathrm{p}_1\mathrm{p}_3(1 - \cos\Theta)$$

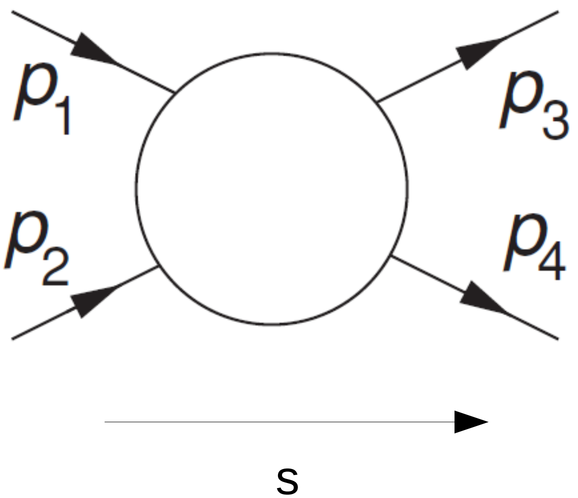
$$\cos\Theta = \frac{\mathbf{p}_1 \cdot \mathbf{p}_3}{\mathrm{p}_1\mathrm{p}_3} \qquad \mathrm{p}_i = |\mathbf{p}_i|$$

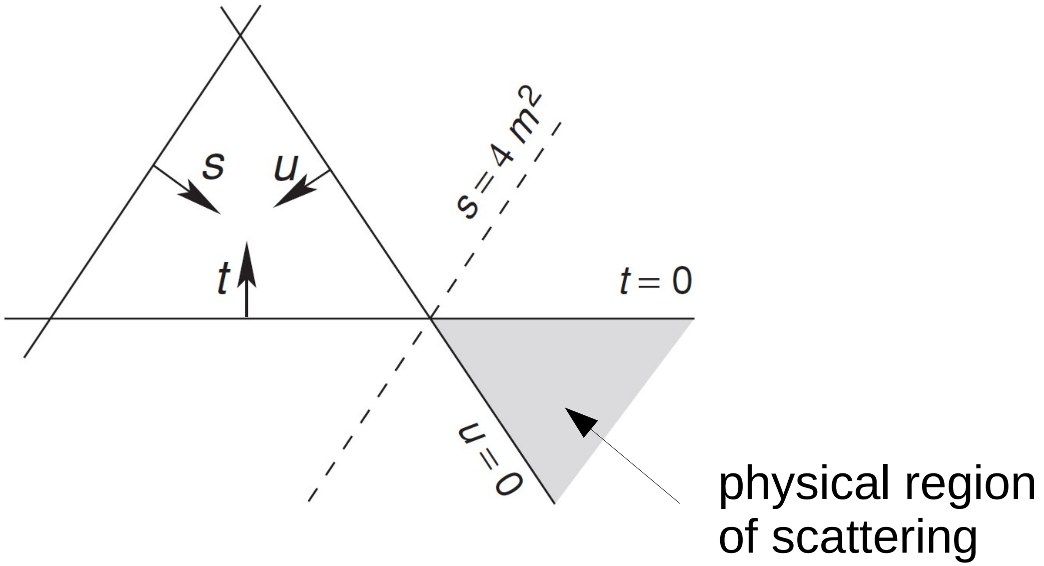
$$\mathrm{p}_1 = \mathrm{p}_2 = p_c \qquad \mathrm{p}_3 = \mathrm{p}_4 = p'_c$$

(continued)

- Set all masses equal (eg: pion-pion scattering). Then:

$$t = -2p_c^2(1 - \cos\Theta_c) \qquad u = -2p_c^2(1 + \cos\Theta_c)$$

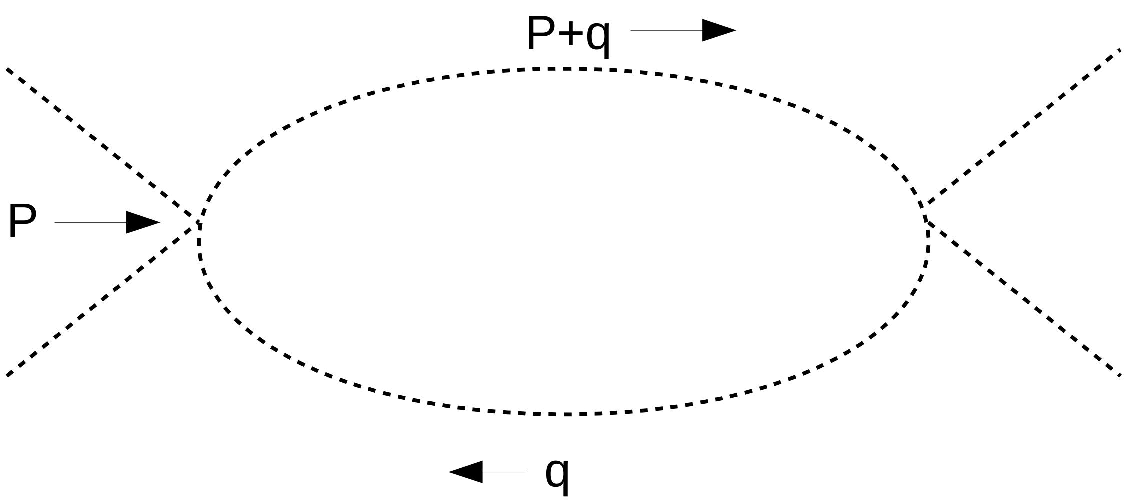




- Physical scattering amplitude is complex.
- Can we see this from a Feynman diagram?

Unitarity

Calculation of 1-loop $\pi\pi$ scattering



$$iG \equiv \int \frac{d^4q}{(2\pi)^4} \frac{i}{(P+q)^2 - m_1^2 + i\epsilon} \frac{i}{q^2 - m_2^2 + i\epsilon}$$

$$= i \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} \frac{1}{s - (\omega_1 + \omega_2)^2 + i\epsilon}$$

(blackboard)

A typical coupled-channel problem

- Pion-nucleon scattering

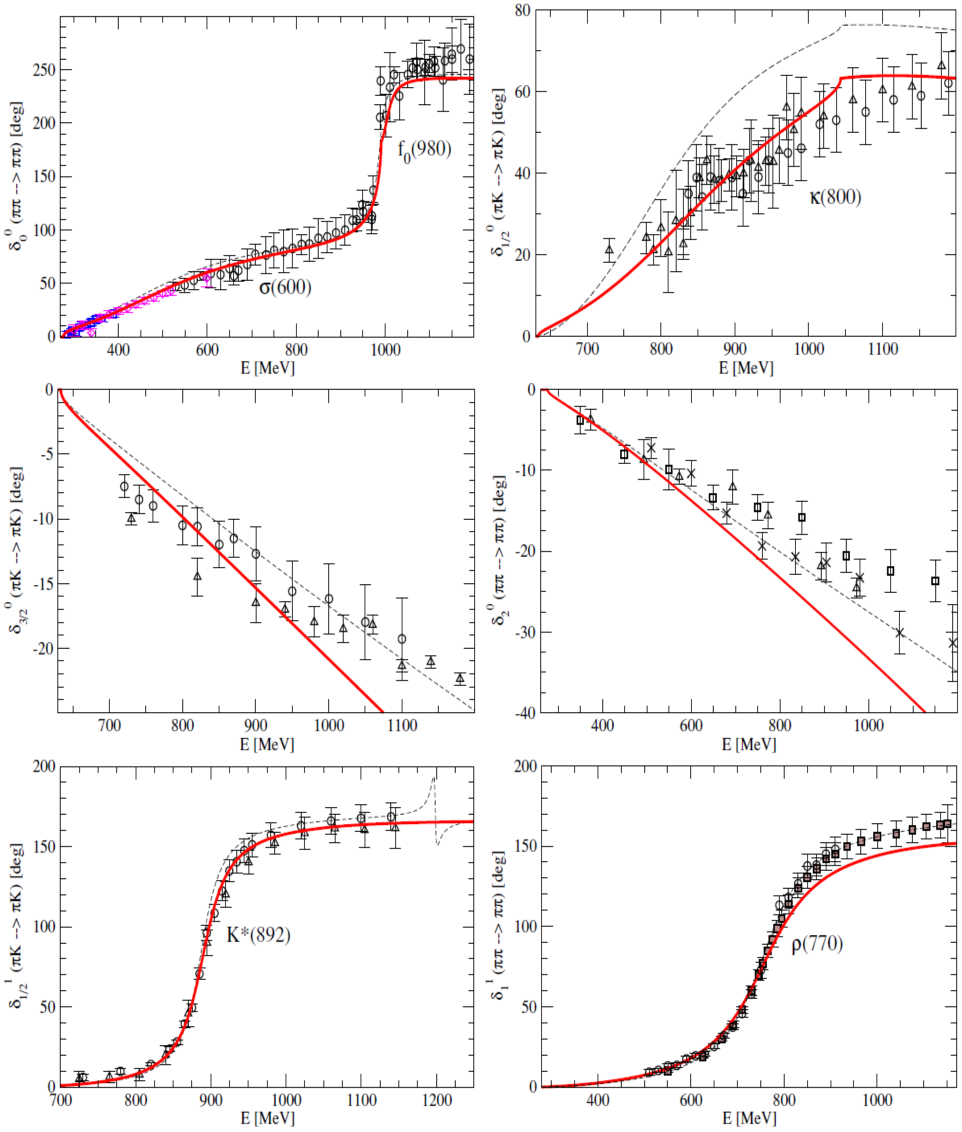
Table 11. Angular momentum structure of the coupled channels in isospin $I = 1/2$ up to $J = 9/2$. The $I = 3/2$ sector is similar up to obvious isospin selection rules.

μ	$J^P =$	$\frac{1}{2}^-$	$\frac{1}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^-$	$\frac{5}{2}^-$	$\frac{5}{2}^+$	$\frac{7}{2}^+$	$\frac{7}{2}^-$	$\frac{9}{2}^-$	$\frac{9}{2}^+$
1	πN	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
2	$\rho N(S = 1/2)$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
3	$\rho N(S = 3/2, \|J - L\| = 1/2)$	–	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
4	$\rho N(S = 3/2, \|J - L\| = 3/2)$	D_{11}	–	F_{13}	S_{13}	G_{15}	P_{15}	H_{17}	D_{17}	I_{19}	F_{19}
5	ηN	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
6	$\pi \Delta(\|J - L\| = 1/2)$	–	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
7	$\pi \Delta(\|J - L\| = 3/2)$	D_{11}	–	F_{13}	S_{13}	G_{15}	P_{15}	H_{17}	D_{17}	I_{19}	F_{19}
8	σN	P_{11}	S_{11}	D_{13}	P_{13}	F_{15}	D_{15}	G_{17}	F_{17}	H_{19}	G_{19}
9	$K \Lambda$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
10	$K \Sigma$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}

[D. Ronchen, M. Doring et al., EPJA 49, 44 (2013)]

Partial wave projection

- Resonances are characterized by a full set of quantum numbers; this full set depends on the participating particles. E.g.: Baryon resonances: J^P (total angular momentum and parity) plus isospin.
- The easiest case is the scattering of two spinless mesons. quantum numbers are: orbital angular momentum L=J and isospin. With $S = e^{2i\delta}$ ("phase shift"):



m_ρ

Partial-wave decomposition

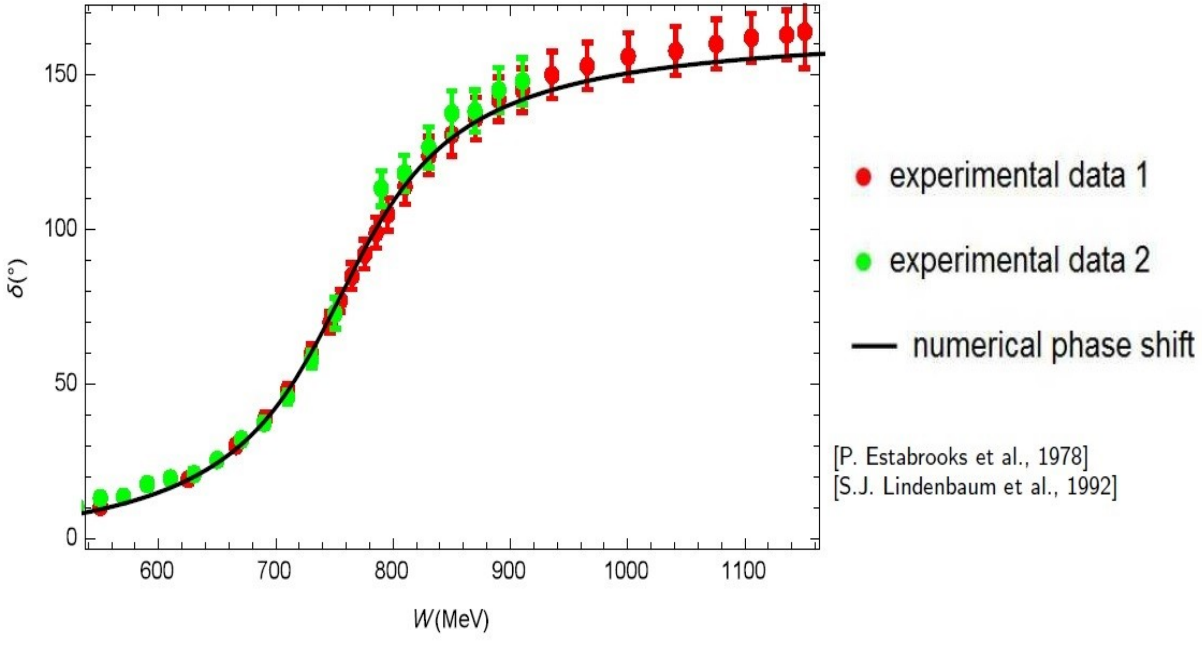
$$T_l^I(s) = \frac{1}{64\pi}\int_{-1}^{1} \mathrm{d}(\cos\theta) P_l(\cos\theta) T^I(s,t,u) \qquad T^I = \sum_{l=0}^{\infty}(2l+1)T_l^I P_l(\cos\theta)$$

phase shift:

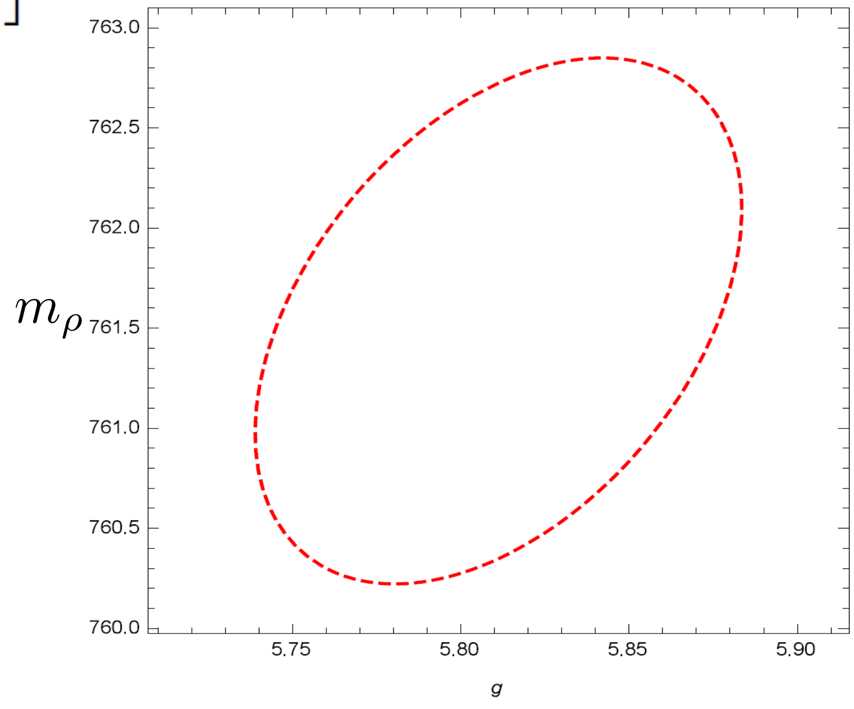
$$T_l^I = \left(\frac{s}{s-4m_\pi^2}\right)^{\frac{1}{2}} e^{i\delta_l^I}\sin\delta_l^I$$

$$\delta_l^I = \frac{1}{2}\arccos\left[\mathrm{Re}\left(1+\frac{2iT_l^I}{\sqrt{\frac{s}{s-4m_\pi^2}}}\right)\right]$$

$$\delta_l^I = \frac{1}{2}\arcsin\left[\mathrm{Im}\left(1+\frac{2iT_l^I}{\sqrt{\frac{s}{s-4m_\pi^2}}}\right)\right]$$



[P. Estabrooks et al., 1978]
[S.J. Lindenbaum et al., 1992]



Summary: Unitarity

- The *S*-matrix is *unitary* (additional explications on blackboard):

$$S\,S^\dagger = 1 \quad \Longrightarrow \quad T_{ab} - T^\dagger_{ab} = i\left(T\,T^\dagger\right)_{ab}$$

- In matrix notation:

$$\frac{1}{i}\left(T_{ab} - T^*_{ba}\right) = \sum_c T_{ac}\,T^*_{cb}$$

$$\frac{1}{[n!]}$$

Intermediate states *c*: Sum over all possible quantum numbers, momenta, and even particle species

→ Concept of coupled channels (blackboard)

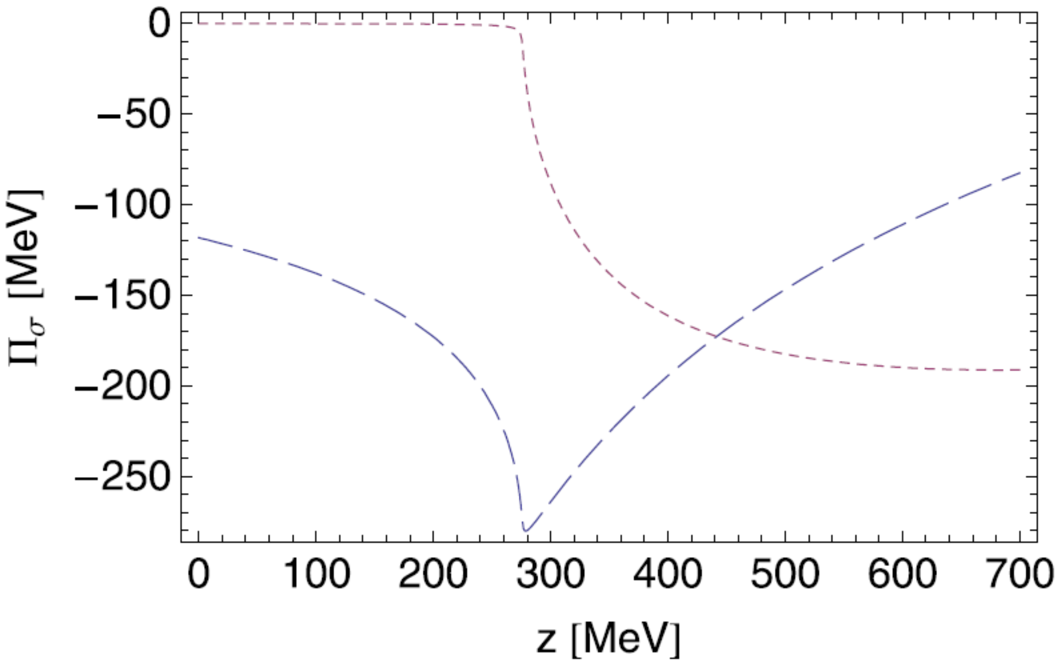
- Time reversal invariance: $T_{ab} = T_{ba}$

$$\frac{1}{i}(T_{ab} - T^*_{ab}) = 2\,\mathrm{Im}\,T_{ab} = \sum_c T_{ac}\,T^*_{bc}$$

Analytic Structure: Riemann sheets

- For a pedagogical introduction, see, e.g. Nuclear Physics A 829 (2009) 170–209
- Consider the loop function:

$$\Pi_\sigma(z,k) = \int_0^\infty q^2\,dq\,\frac{(v^{\sigma\pi\pi}(q,k))^2}{z - 2\sqrt{q^2+m_\pi^2} + i\epsilon}$$

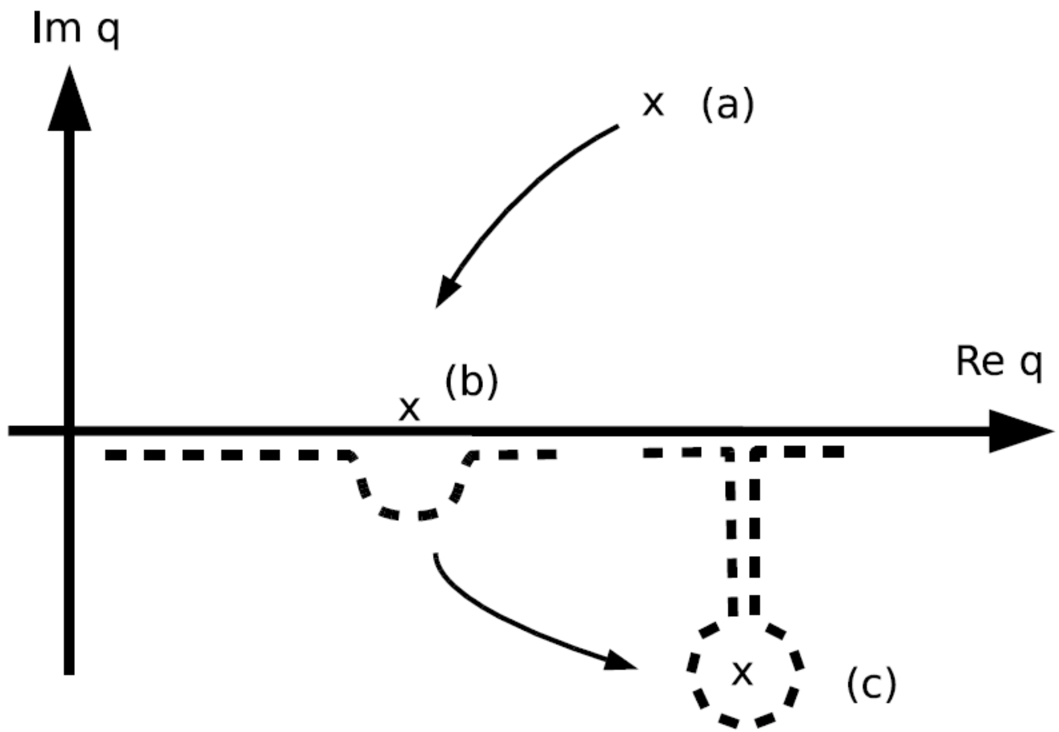


Integration paths

- Imaginary part: $\mathrm{Im}\,\Pi_\sigma = -\frac{\pi\, q_{\mathrm{on}}^{>}\, E_{\mathrm{on}}^{(1)}\, E_{\mathrm{on}}^{(2)}}{z}\, v^2(q_{\mathrm{on}}^{>}, k)$
- pole in the integrand at

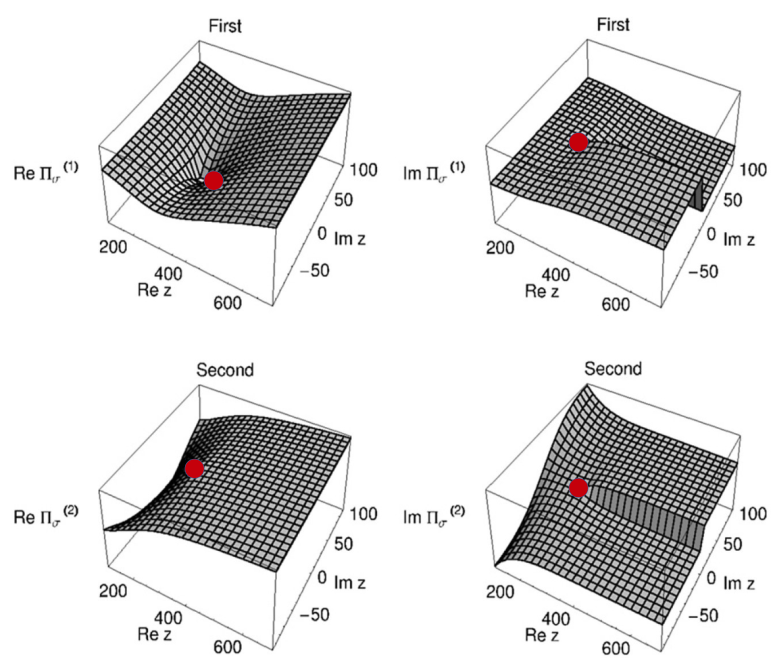
$$q_{\mathrm{on}} = \frac{1}{2z}\sqrt{\left(z^2-(m_1-m_2)^2\right)\left(z^2-(m_1+m_2)^2\right)}$$

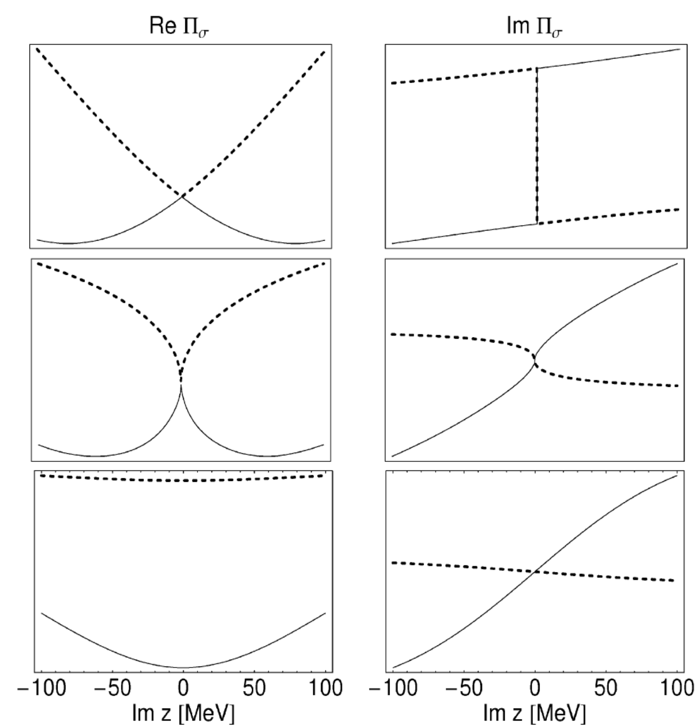
- Integratior paths for $\mathrm{Im}\,z > 0, = 0, < 0$ (a), (b), (c)



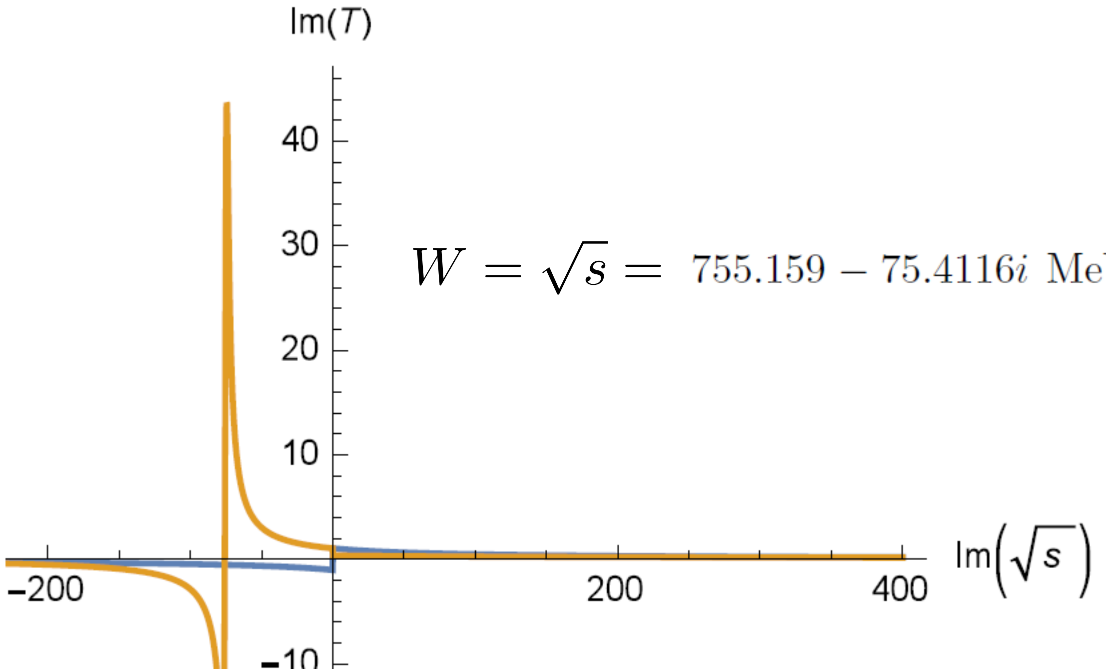
Analytic continuation

- Obviously, the loop has a discontinuity at Im z=0 starting at $z = m_1 + m_2$
- We can analytically continue (avoiding the discontinuity) by path deformation; equivalent to adding the imaginary part twice; latter given by residue (check)!
- Resulting structure: Two Riemann sheets and one branch point at threshold:





The rho pole in the complex plane



Alternative definition of the coupling constant g through the pole residue:

$$g^2_{\text{meson-meson}} = -16\pi \lim_{s \to s_{\text{pole}}} (s - s_{\text{pole}}) T(s) \frac{3}{4Q^2}$$

$$g_{\text{meson-meson}} = 5.99392 - 0.792795i, \; |g_{\text{meson-meson}}| = 6.04612$$

compared to g from Lagrangian: $g = 5.81118$

3-particles: Unitarity

$$T_{fi} - T^{\dagger}_{fi} = i \sum_n d\Omega_n \, T_{fn} T^{\dagger}_{ni}$$

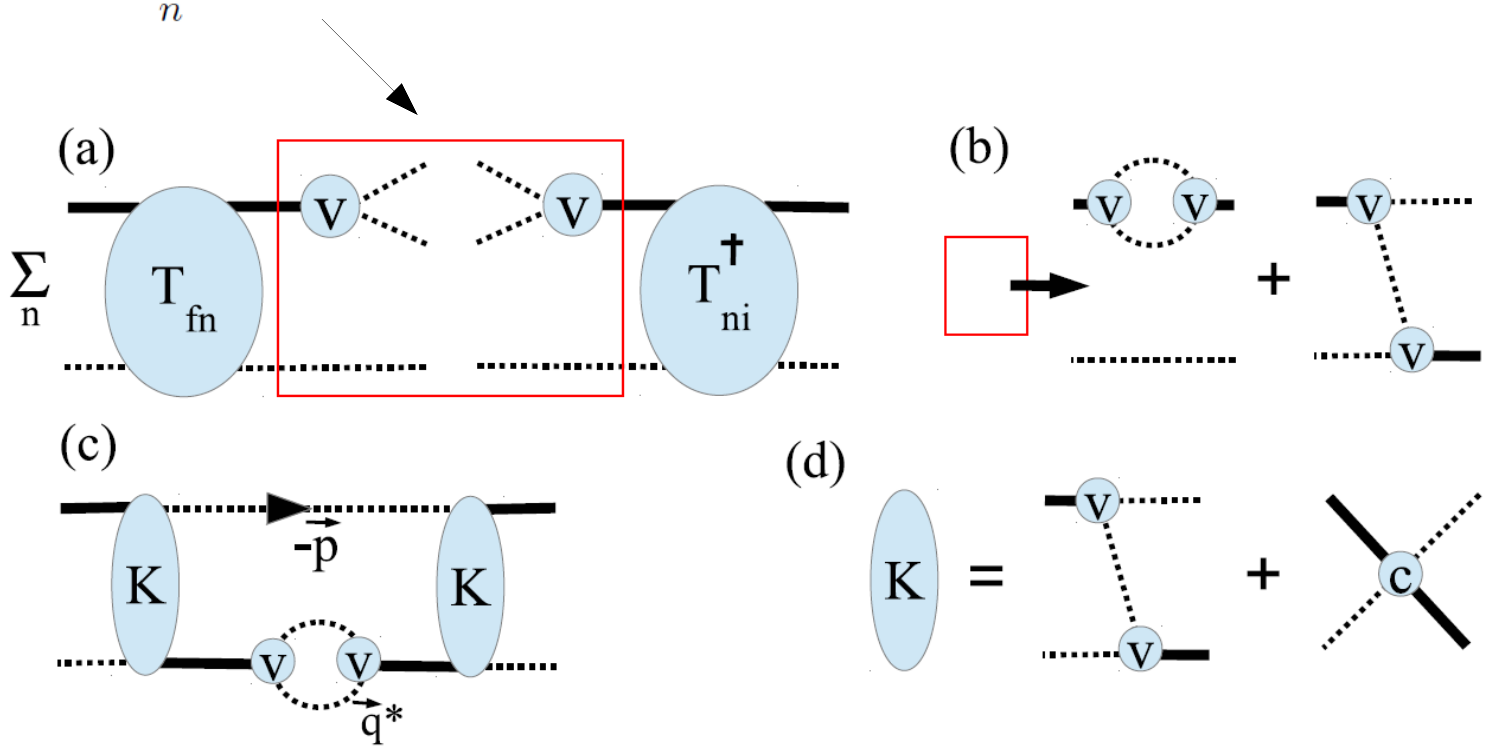


Figure 1: (a) Unitarity relation for three particles [right-hand side of Eq. (1)]. The elementary particles (e.g., pions) are shown with the dashed lines, the auxiliary fields/isobars (e.g., ρ) are represented by the solid lines. (b) There are two distinct ways to combine the three particles in the sum. This leads to resummed self-energy contributions and exchange processes in the amplitude. (c) Amplitude at one loop (only one term of the resummed self-energy is indicated). (d) Contact interactions c can be added without spoiling three-body unitarity.

3-particles: Branch points

- (Derivation: blackboard)
- **Test:**

  Fit with a model **A**

  with only poles but

  without ρN branch points

  (solid lines)

  to an amplitude **B** with

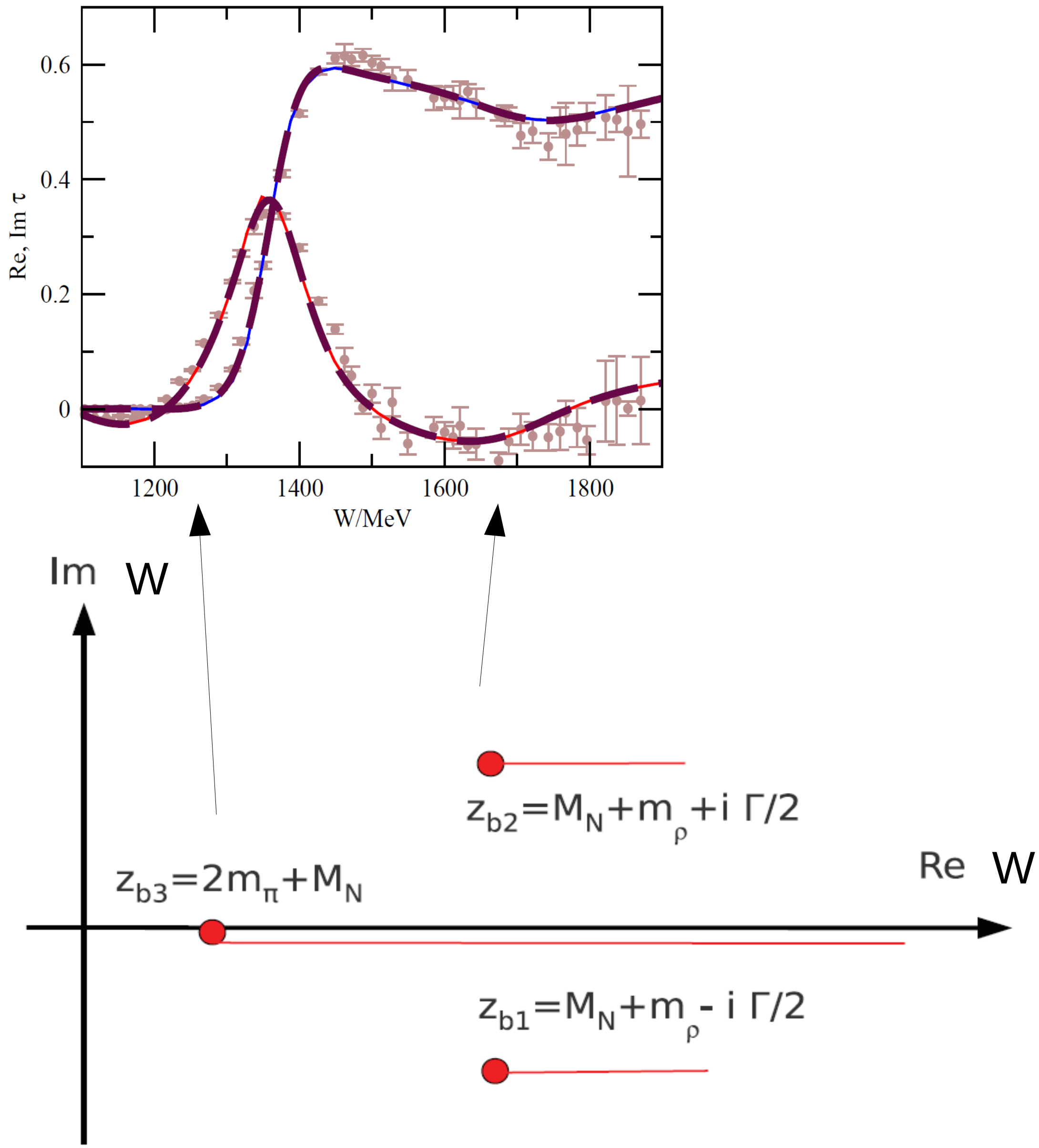
  ONLY ρN branch points

  (dashed lines)
- Model **A** "finds" a pole

  at 1698-130i MeV
- Implementation of correct

  analytic structure crucial

[PRC 84, 015205]



A typical coupled-channel problem

- Pion-nucleon scattering

Table 11. Angular momentum structure of the coupled channels in isospin $I = 1/2$ up to $J = 9/2$. The $I = 3/2$ sector is similar up to obvious isospin selection rules.

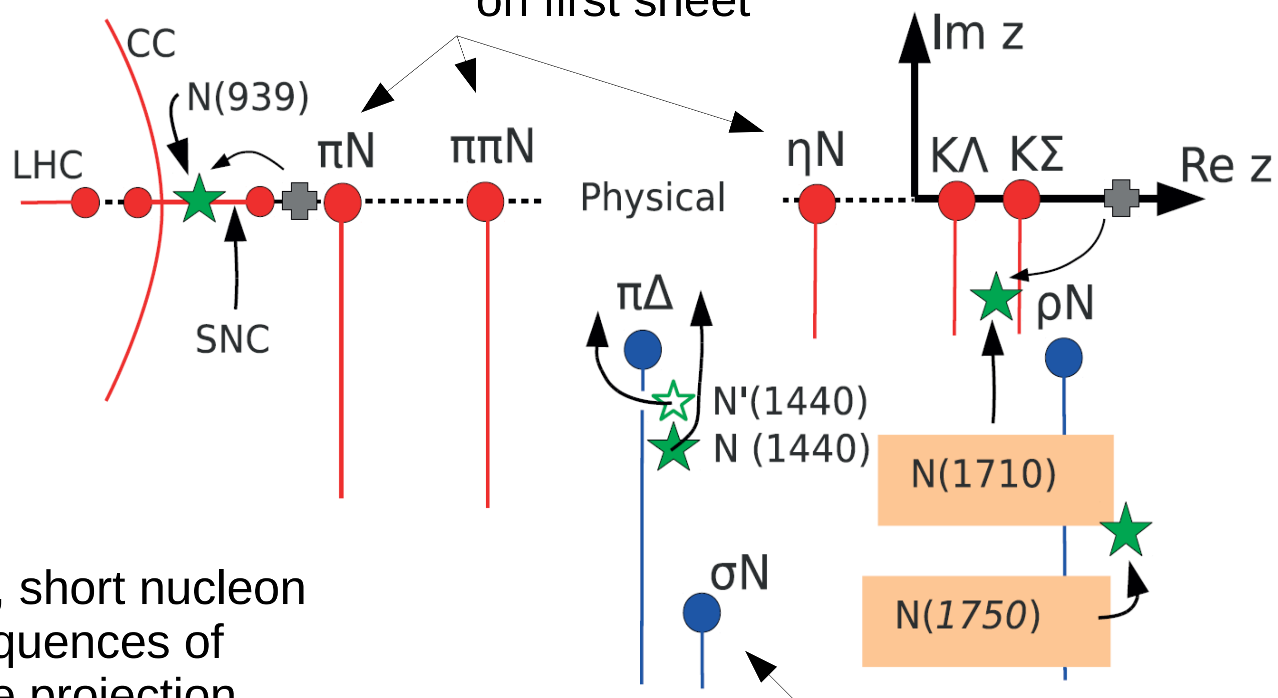
μ	$J^P =$	$\frac{1}{2}^-$	$\frac{1}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^-$	$\frac{5}{2}^-$	$\frac{5}{2}^+$	$\frac{7}{2}^+$	$\frac{7}{2}^-$	$\frac{9}{2}^-$	$\frac{9}{2}^+$
1	πN	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
2	$\rho N(S = 1/2)$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
3	$\rho N(S = 3/2, \lvert J - L\rvert = 1/2)$	–	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
4	$\rho N(S = 3/2, \lvert J - L\rvert = 3/2)$	D_{11}	–	F_{13}	S_{13}	G_{15}	P_{15}	H_{17}	D_{17}	I_{19}	F_{19}
5	ηN	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
6	$\pi\Delta(\lvert J - L\rvert = 1/2)$	–	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
7	$\pi\Delta(\lvert J - L\rvert = 3/2)$	D_{11}	–	F_{13}	S_{13}	G_{15}	P_{15}	H_{17}	D_{17}	I_{19}	F_{19}
8	σN	P_{11}	S_{11}	D_{13}	P_{13}	F_{15}	D_{15}	G_{17}	F_{17}	H_{19}	G_{19}
9	$K\Lambda$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}
10	$K\Sigma$	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}	H_{19}

[D. Ronchen, M. Doring et al., EPJA 49, 44 (2013)]

Amplitude structure – Partial waves in πN

- This structure translates from the loop function/self energy to the entire amplitude *T*; but *V* in *T=V+VGT* has also non-analyticities; general structure is complicated, e.g. pion-nucleon scattering:

channel openings:
branch points
on first sheet

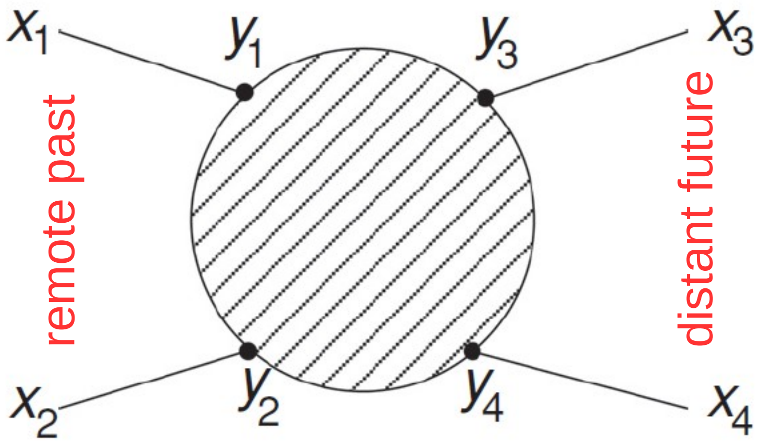


circular cut, short nucleon cut: Consequences of partial wave projection, NOT present in full amplitude

Branch points of unstable particles lie on second sheet!

Causality and Analyticity

- 4-point Green function $A(x_1, x_2; x_3, x_4)$

$A(x_1, x_2; x_3, x_4) =$ 

$$= \int f(y_1, y_2, y_3, y_4) \left\{ \prod_{i=1}^{4} D(y_i - x_i)\, d^4 y_i \right\}$$

- *D(y-x)*: free particle propagation

$$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{dp_0}{2\pi i} \frac{\exp\{-ip^\mu (y-x)_\mu\}}{m^2 - p^2 - i\epsilon}$$

solve 0-integration

- As $y_0 > x_0$, pole at $p_0 = \sqrt{m^2 + \mathbf{p}^2}$

$$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\exp\{-ip^\mu(y-x)_\mu\}}{2p_0}$$
$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \psi_\mathbf{p}(y) \cdot \psi^*_\mathbf{p}(x), \qquad y_0 > x_0$$

- while for final state $x_{03} > y_{03}, x_{04} > y_{04}$

$$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \psi_\mathbf{p}(x) \cdot \psi^*_\mathbf{p}(y), \qquad x_0 > y_0$$

- Truncates amplitude *f* gets multiplied by product of wave functions.

Amplitude in momentum space

- Fourier transform of *f*:

$$\mathcal{M}(p_i) = \int f(y_1, y_2, y_3, y_4)\, \mathrm{e}^{-i(p_1 y_1 + p_2 y_2) + i(p_3 y_3 + p_4 y_4)} \prod d^4 y_i$$

- Make it simple:
  - *forward scattering* $p_1 \approx p_3, p_2 \approx p_4$
  - Solve some integrals → only dependence on relative positions, here $y_{13} = y_1 - y_3$ chosen:

$$\mathcal{M} \Longrightarrow (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \int \mathrm{e}^{ip_1(y_3 - y_1)} f(y_{13}; p_2) d^4 y_{13}$$

- Forward scattering → Only dependence on one variable

(continued)

- The amplitude is proportional to the absorption of a particle in y_1 and creation in y_3 (and reversely for anti-particle):

$$f(y_3, y_1) \propto \langle T\, \psi(y_3)\bar{\psi}(y_1)\rangle \qquad \Delta y^\mu = y_3^\mu - y_1^\mu$$

$$\equiv \vartheta(\Delta y_0)\cdot \psi(y_3)\bar{\psi}(y_1) \;\pm\; \vartheta(-\Delta y_0)\cdot \bar{\psi}(y_1)\psi(y_3)$$

$$= \vartheta(\Delta y_0)\left[\psi(y_3)\bar{\psi}(y_1) \mp \bar{\psi}(y_1)\psi(y_3)\right] \pm \bar{\psi}(y_1)\psi(y_3)$$

(compare to the time evolution operator U in QM which is a time-ordered product; the S-matrix is actually a time-evolution operator)

- Consider now a space-like interval $(\Delta y)^2 < 0$
- The operators $\psi(y_3)\bar{\psi}(y_1)$ have to commute; otherwise a person at y_3 could tell what was measured at y_1 → Causality!
- Then: $f(y_3, y_1) \propto \vartheta(\Delta y_0)\vartheta((\Delta y)^2)\cdot f_1 \pm \bar{\psi}(y_1)\psi(y_3)$
- Insert unity in the last term:

$$\langle 0|\, \bar{\psi}(y_1)\psi(y_3)\,|0\rangle = \sum_n \langle 0|\, \bar{\psi}(y_1)|n\rangle \cdot \langle n|\psi(y_3)|0\rangle = \sum_n |C_n|^2\, \mathrm{e}^{-iP_n(y_1-y_3)}$$

(continued)

- We still have to integrate over *y* to get *M* (see previous slides):

$$\sum_n |C_n^2| \int d^4y_{31}\ \mathrm{e}^{ip_1 y_{31}} \cdot \mathrm{e}^{iP_n y_{31}} \propto \delta(p_{0,1} + P_{0,n}) \ = \ 0$$

- This has to be zero because all incoming, outgoing, intermediate particles have positive energy, e.g., $P_{0,n} > 0$
- Finally, as $p_1 y \equiv E_1 t - \mathbf{p}_1 \cdot \mathbf{y} = E_1 \cdot (t - v_1 z)$

$$\mathcal{M}(E_1) = \int d^4y\, f_1(y) \cdot \vartheta(y_0)\vartheta(y_\mu^2)\, \mathrm{e}^{ip_1 y} = \int d^3\mathbf{y} \int_{\sqrt{\mathbf{y}^2}}^{\infty} dt\, \mathrm{e}^{iE_1(t - v_1 z)} f_1(y)$$

- Make use of all delta-functions →

$$t > 0, \quad t > \sqrt{z^2 + \boldsymbol{\rho}_\perp^2} \geq |z| > |v_1 z| \Longrightarrow (t - v_1 z) > 0$$

- If $\mathrm{Im}\, E_1 > 0$ and *f* increases less than expon., M converges in the upper half plane.

(continued)

- Implies the so-called *polynomial boundary* for *M(s)*

$$|\mathcal{M}(s)| < |s|^N$$

- Absolut converging integral → Integration and differentiation can be interchanged.
- Cauchy relations:

$$u = u(x,y),\ v = v(x,y),\ z = x + iy \rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

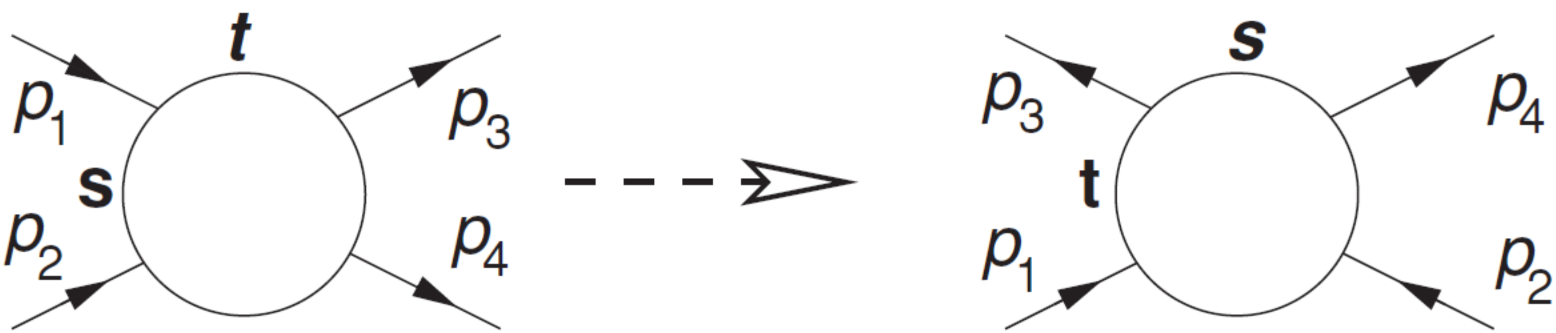
- hold in the upper half plane with

$$u = \mathrm{Re}\,\mathcal{M}, v = \mathrm{Im}\,\mathcal{M},\ z = E_1$$

- Cauchy relations fulfilled ↔ function analytic.

Crossing Symmetry

- Consider another process by turning the scattering around:



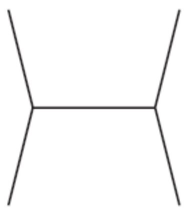
$12 \to 34$ $\quad\quad$ $1\bar{3} \to \bar{2}4$

- negative 0-components appear: $p_{30} \leq -m_3$ and $p_{20} \leq -m_2$
- interpretation of crossed process: anti-particle with $\bar{p}_3 = -p_3$
- Crossed diagram describes another process; so called *t*-channel reaction:

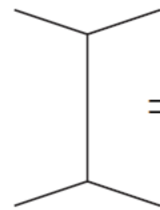
$$t = (p_1 - p_3)^2 = (p_1 + \bar{p}_3)^2 \geq (m_1 + m_3)^2$$

Crossed processes in λ^3

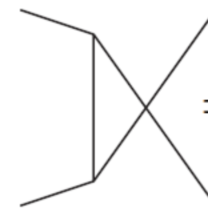
- s, t, and u-channel processes in λ^3 theory:



$$= \frac{\lambda^2}{m^2 - s},$$



$$= \frac{\lambda^2}{m^2 - t},$$



$$= \frac{\lambda^2}{m^2 - u}$$

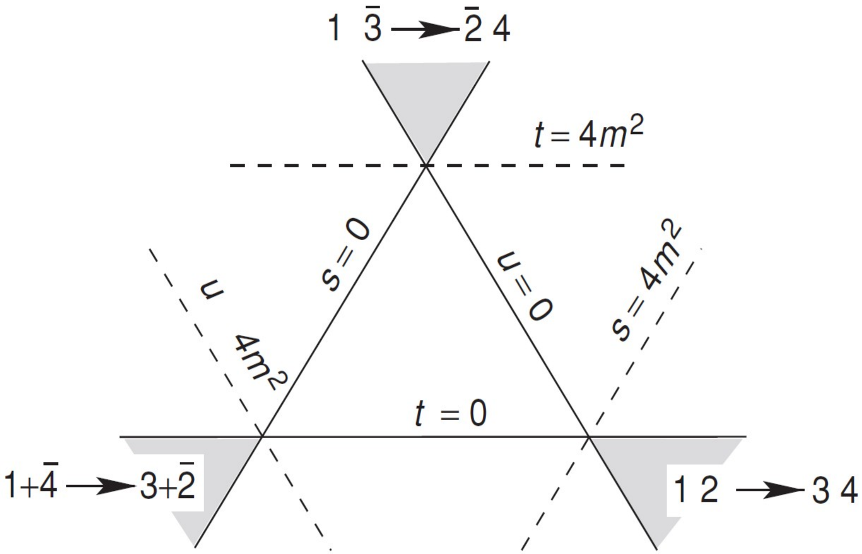
u-channel reactions

- Analogously, if $p_{40} \leq -m_4,\ p_{20} \leq -m_2$: $\quad 1 + \bar{4} \to 3 + \bar{2}$

$$u = (p_1 - p_4)^2 = (p_1 + \bar{p}_4)^2 \geq (m_1 + m_4)^2$$

- In summary:

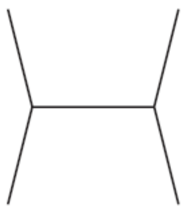
$$\begin{aligned} s\text{-channel}: &\quad 1+2 \to 3+4, \quad s = (p_1+p_2)^2 \geq (m_1+m_2)^2; \\ t\text{-channel}: &\quad 1+\bar{3} \to \bar{2}+4, \quad t = (p_1+\bar{p}_3)^2 \geq (m_1+m_3)^2; \\ u\text{-channel}: &\quad 1+\bar{4} \to 3+\bar{2}, \quad u = (p_1+\bar{p}_4)^2 \geq (m_1+m_4)^2. \end{aligned}$$



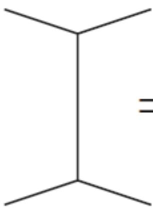
- There are 3 unitarity relations:
  - s-channel unitarity
  - t-channel unitarity
  - u-channel unitarity

Analytic structure in the Mandelstam plane

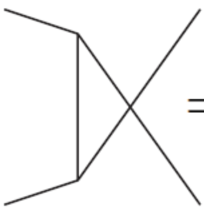
- Again, s-, t-, and u-channel processes:



$$= \frac{\lambda^2}{m^2 - s},$$



$$= \frac{\lambda^2}{m^2 - t},$$

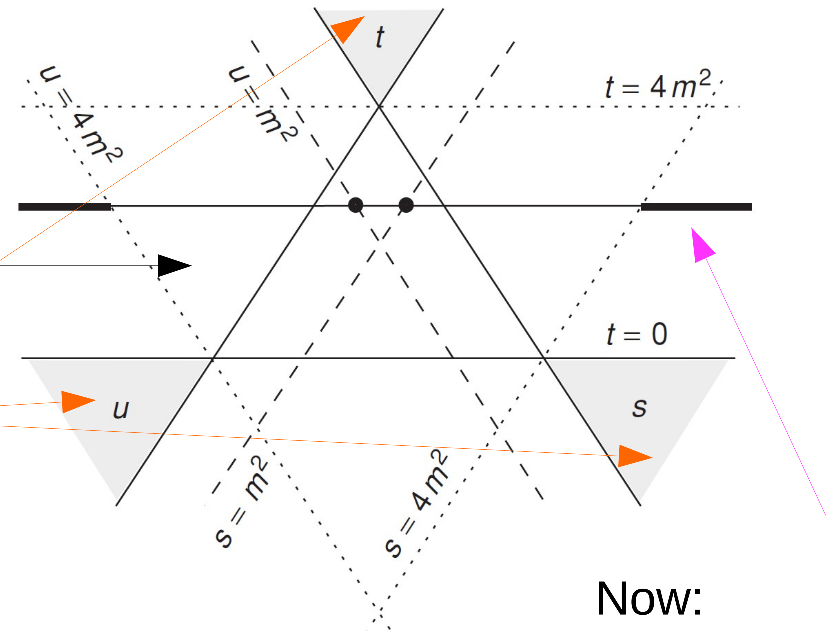


$$= \frac{\lambda^2}{m^2 - u}$$

- Induce poles in the amplitude, at position of physical particle mass (which is NOT *m*).

Amplitude real inside dotted triangle

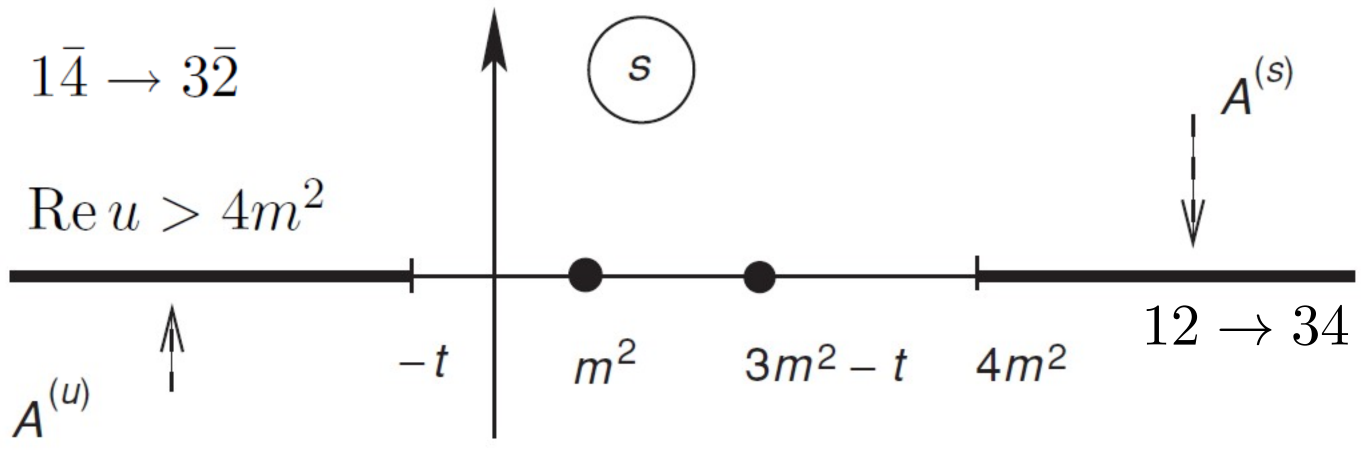
Physical regions



Now:
fix t and consider amplitude as *f(s)*

Left- and right-hand cut

- The <u>only</u> non-analyticities on the first Riemann sheet:



Dispersive representation of the amplitude

- Cauchy's Theorem:

$$\int_C \frac{dz}{2\pi i} \frac{A(z)}{z-s} = A(s)$$

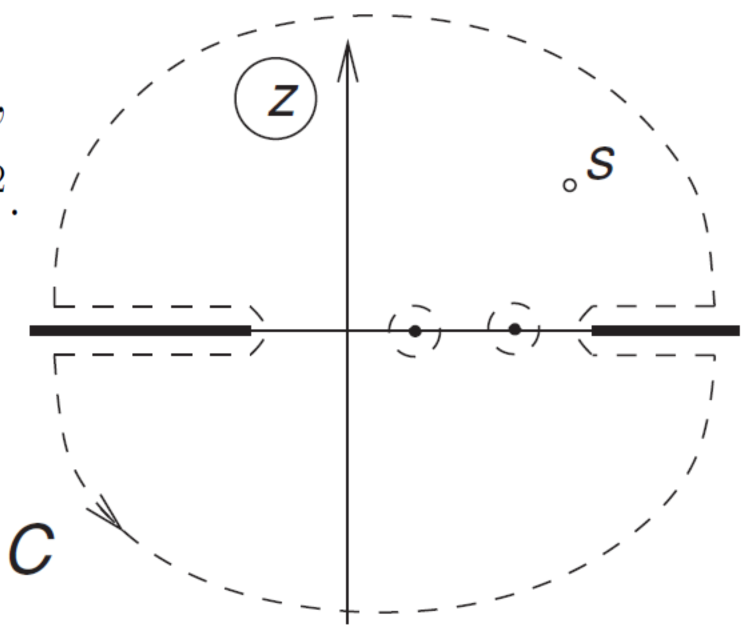
$\lambda^2/(m^2-t)$ does not fall with s → once-subtracted dispersion relation:

$$A(s) - A(0) = \int_C \frac{dz}{2\pi i} \left[\frac{A(z)}{z-s} - \frac{A(z)}{z} \right] = \frac{s}{\pi} \int_C \frac{dz}{2i} \frac{A(z)}{z(z-s)}$$

$$\mathrm{Im}_s\, A \equiv \frac{1}{2i}\left[A(s+i0,t) - A(s-i0,t)\right], \quad s > 4m^2,$$

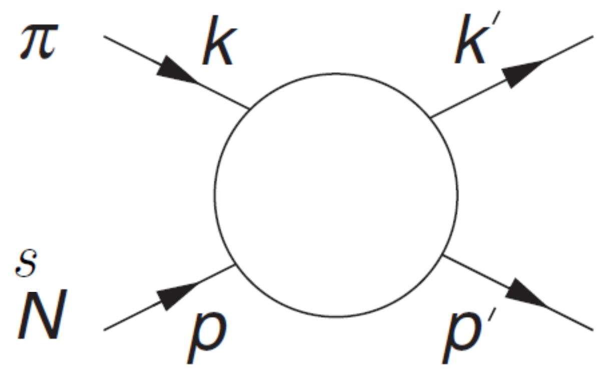
$$\mathrm{Im}_u\, A \equiv \frac{1}{2i}\left[A(u+i0,t) - A(u-i0,t)\right], \quad u > 4m^2.$$

→ Simplify the expression!



Example: Pion-nucleon scattering

- forward scattering *t=0*:

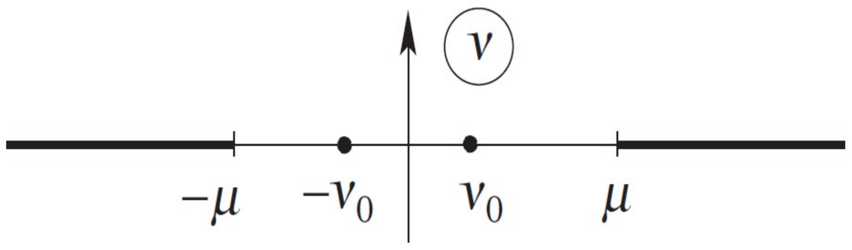


$$s = (p+k)^2 = M^2 + \mu^2 + 2M\nu,$$

$$u = (p-k')^2 = M^2 + \mu^2 - 2M\nu = 2(M^2+\mu^2) - s$$

ν Energy of the pion in nucleon rest frame

$$\nu = \frac{s-u}{2M} = \frac{s-(M^2+\mu^2)}{M}$$



$$f(\nu) = \frac{r}{\nu_0 - \nu} + \frac{1}{\pi}\int_{\mu}^{\infty} \frac{d\nu' \,\mathrm{Im}\, f(\nu')}{\nu' - \nu}$$

$$+ \frac{r}{\nu_0 + \nu} + \frac{1}{\pi}\int_{-\mu}^{-\infty} \frac{d\nu' \,\mathrm{Im}\, f(\nu')}{\nu' - \nu}$$

$$A = \overline{\mathbf{U}}(p')\phi_\alpha(k')\big(f_+(\nu)\delta_{\alpha\beta}\cdot \mathbf{I} + f_-(\nu)\varepsilon_{\alpha\beta\gamma}\cdot \boldsymbol{\tau}_\gamma\big)\phi_\beta(k)\mathbf{U}(p)$$

As $f(-\nu) = f(\nu)$

for f= f_+ →

$$f(\nu) = f(0) + \frac{2r}{\nu_0}\frac{\nu^2}{\nu_0^2 - \nu^2} + \frac{\nu^2}{\pi}\int_{\mu}^{\infty} \frac{d\nu'^2}{\nu'^2}\frac{\mathrm{Im}\, f(\nu')}{(\nu'^2 - \nu^2)}$$

END

Pion-Pion Scattering via the rho-Meson

Ingredients

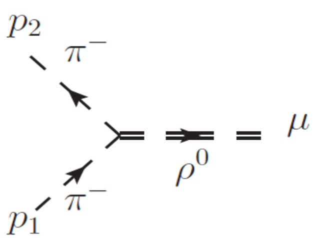
$$\mathscr{L}_\rho = -\frac{1}{8}\mathrm{Tr}(\rho_{\mu\nu}\rho^{\mu\nu}) + \frac{1}{4}m_\rho^2\mathrm{Tr}(\rho_\mu\rho^\mu)$$

$$\begin{aligned} \rho_{\mu\nu} &= \partial_\mu\rho_\nu - \partial_\nu\rho_\mu \\ \rho &= \sum_{i=1}^{3}\rho_i\tau_i = \begin{pmatrix} \rho_\mu^3 & \rho_\mu^1 - i\rho_\mu^2 \\ \rho_\mu^1 + i\rho_\mu^2 & -\rho_\mu^3 \end{pmatrix} \\ &= \begin{pmatrix} \rho_\mu^0 & \sqrt{2}\rho_\mu^+ \\ \sqrt{2}\rho_\mu^- & -\rho_\mu^0 \end{pmatrix} \end{aligned}$$

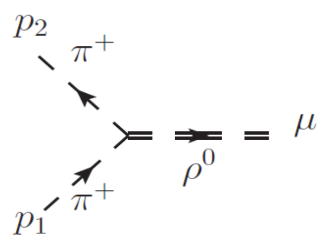
$$\begin{aligned} \mathscr{L} &= \frac{1}{4}\mathrm{Tr}(D_\mu\pi D^\mu\pi - m_\pi^2\pi^2) - \frac{1}{8}\mathrm{Tr}(\rho_{\mu\nu}\rho^{\mu\nu}) + \frac{1}{4}m_\rho^2\mathrm{Tr}(\rho_\mu\rho^\mu) \\ &= \mathscr{L}_0 + \mathscr{L}_{\mathrm{int}} \qquad\qquad D_\mu\pi = \partial_\mu\pi - i\tfrac{g}{2}[\rho_\mu,\pi] \end{aligned}$$

$$\begin{aligned} \mathscr{L}_0 &= \frac{1}{4}\mathrm{Tr}(\partial_\mu\pi\partial^\mu\pi - m_\pi^2\pi^2) - \frac{1}{8}\mathrm{Tr}(\rho_{\mu\nu}\rho^{\mu\nu}) + \frac{1}{4}m_\rho^2\mathrm{Tr}(\rho_\mu\rho^\mu) \\ \mathscr{L}_{\mathrm{int}} &= \frac{ig}{4}\mathrm{Tr}(\rho_\mu[\partial^\mu\pi,\pi]) - \frac{g^2}{16}\mathrm{Tr}\left([\rho_\mu,\pi][\rho^\mu,\pi]\right) \\ \mathscr{L}_{\rho\pi\pi} &= \frac{ig}{4}\mathrm{Tr}(\rho_\mu[\partial^\mu\pi,\pi]) \\ &= ig\big[\rho_\mu^0\big((\partial^\mu\pi^+)\pi^- - (\partial^\mu\pi^-)\pi^+\big) + \rho_\mu^-\big((\partial^\mu\pi^0)\pi^+ - (\partial^\mu\pi^+)\pi^0\big) \\ &\quad +\rho_\mu^+\big((\partial^\mu\pi^-)\pi^0 - (\partial^\mu\pi^0)\pi^-\big)\big] \end{aligned}$$

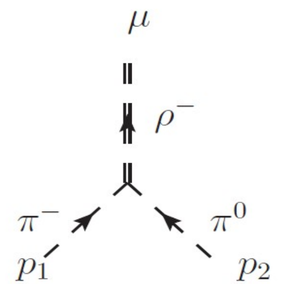
Feynman Rules



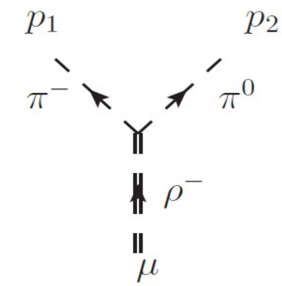
$= -ig(p_1 + p_2)^\mu$



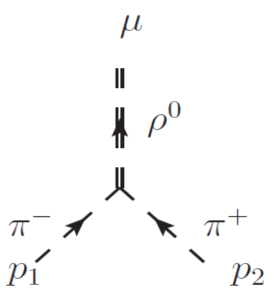
$= -ig(-p_1 - p_2)^\mu$



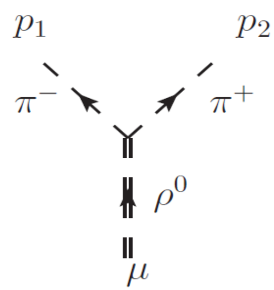
$= -ig(-p_1 + p_2)^\mu$



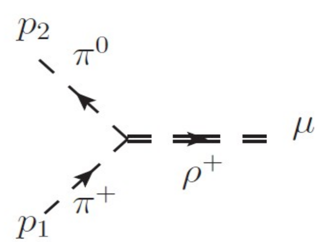
$= -ig(-p_1 + p_2)^\mu$



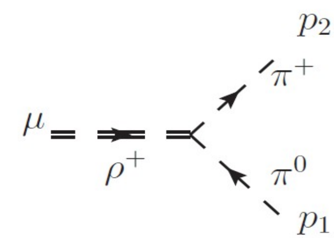
$= -ig(p_1 - p_2)^\mu$



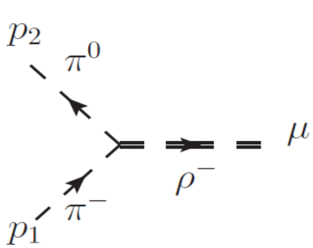
$= -ig(p_1 - p_2)^\mu$



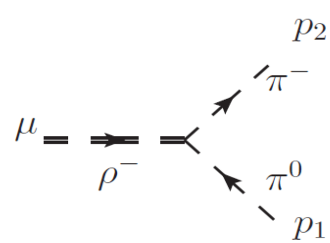
$= -ig(p_1 + p_2)^\mu$



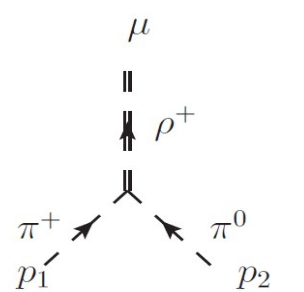
$= -ig(p_1 + p_2)^\mu$



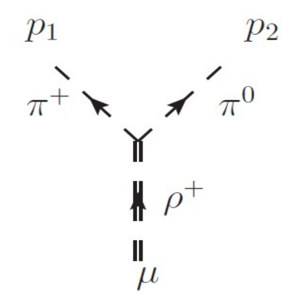
$= -ig(-p_1 - p_2)^\mu$



$= -ig(-p_1 - p_2)^\mu$



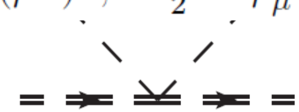
$= -ig(p_1 - p_2)^\mu$



$= -ig(p_1 - p_2)^\mu$

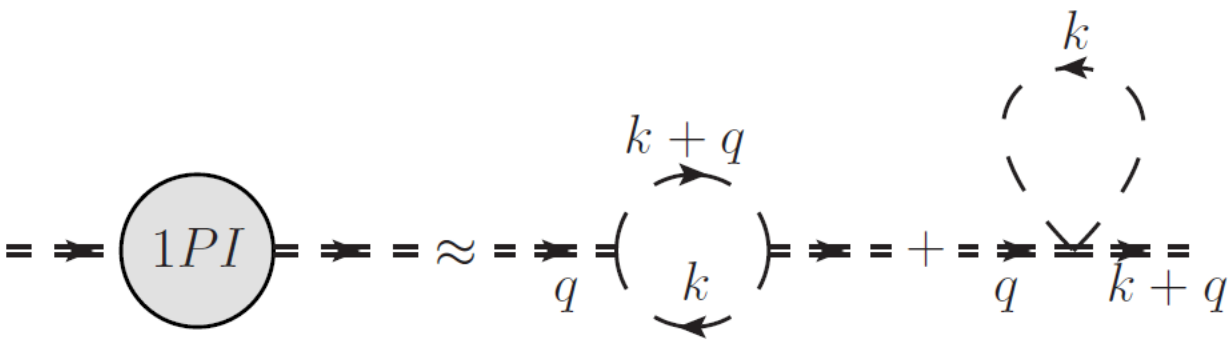
$\mathscr{L}_{\rho\rho\pi\pi}$: $-\frac{g^2}{2}\pi^+\rho^-_\mu\pi^+(\rho^-)^\mu$, $-\frac{g^2}{2}\pi^-\rho^+_\mu\pi^-(\rho^+)^\mu$, $\frac{g^2}{2}\pi^0\rho^+_\mu\pi^0(\rho^-)^\mu$ $\frac{g^2}{2}\pi^0\rho^-_\mu\pi^0(\rho^+)^\mu$

AND



$= ig^{\mu\nu}$

Self Energy



$$
\begin{aligned}
i\Pi^{\mu\nu}(q) &= (-ig)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}(2k+q)^\mu \frac{i}{k^2-m_\pi^2}\frac{i}{(k+q)^2-m_\pi^2}(2k+q)^\nu + 2ig\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{ig^{\mu\nu}}{k^2-m_\pi^2} \\
&= g^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(2k+q)^\mu(2k+q)^\nu - 2g^{\mu\nu}[(k+q)^2-m_\pi^2]}{(k^2-m_\pi^2)[(k+q)^2-m_\pi^2]} \\
&= g^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1 \mathrm{d}x \frac{(2k+q)^\mu(2k+q)^\nu - 2g^{\mu\nu}[(k+q)^2-m_\pi^2]}{\{(k^2-m_\pi^2)(1-x)+[(k+q)^2-m_\pi^2]x\}^2} \\
&= g^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1 \mathrm{d}x \frac{(2k+q)^\mu(2k+q)^\nu - 2g^{\mu\nu}[(k+q)^2-m_\pi^2]}{[k^2-m_\pi^2+2kqx+q^2x]^2} \\
&= g^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1 \mathrm{d}x \frac{(2k+q)^\mu(2k+q)^\nu - 2g^{\mu\nu}[(k+q)^2-m_\pi^2]}{[(k+xq)^2-x(x-1)q^2-m_\pi^2]^2} \\
&= g^2 \int \frac{\mathrm{d}^4l}{(2\pi)^4}\int_0^1 \mathrm{d}x \frac{(2l-2xq+q)^\mu(2l-2xq+q)^\nu - 2g^{\mu\nu}[(l+(1-x)q)^2-m_\pi^2]}{(l^2-\Delta)^2}
\end{aligned}
$$

where we suppose $l = k + xq$, $\Delta = x(x-1)q^2 + m_\pi^2$.

Self energy (continued)

- Wick rotation:

$$l^0 = il_E^0, \vec{l} = \vec{l}_E$$

$$\begin{aligned}
i\Pi^{\mu\nu}(q) &= g^2 \int \frac{\mathrm{d}^d l}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{4l^\mu l^\nu + (1-2x)^2 q^\mu q^\nu - 2g^{\mu\nu}[l^2 + (1-x)^2 q^2 - m_\pi^2]}{(l^2-\Delta)^2} \\
&= g^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} i \frac{\frac{4}{d}(-l_E^2) g^{\mu\nu} + (1-2x)^2 q^\mu q^\nu - 2g^{\mu\nu}[-l_E^2 + (1-x)^2 q^2 - m_\pi^2]}{(-1)^2 (l_E^2+\Delta)^2} \\
&= g^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} i \frac{\left(-\frac{4}{d}+2\right) l_E^2 g^{\mu\nu} + (1-2x)^2 q^\mu q^\nu - 2g^{\mu\nu}(1-x)^2 q^2 + 2g^{\mu\nu} m_\pi^2}{(l_E^2+\Delta)^2} \\
&= ig^2 \int_0^1 \mathrm{d}x \left\{ \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{d}{2} \frac{\Gamma(1-\frac{d}{2})}{\Gamma(2)} \frac{\left(2-\frac{4}{d}\right) g^{\mu\nu}}{\Delta^{1-\frac{d}{2}}} \right. \\
&\quad \left. + \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \frac{1}{\Delta^{2-\frac{d}{2}}} [(1-2x)^2 q^\mu q^\nu - 2g^{\mu\nu}(1-x)^2 q^2 + 2g^{\mu\nu} m_\pi^2] \right\} \\
&= ig^2 \int_0^1 \mathrm{d}x \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-\frac{d}{2}}} [-2\Delta g^{\mu\nu} + (1-2x)^2 q^\mu q^\nu - 2g^{\mu\nu}(1-x)^2 q^2 + 2g^{\mu\nu} m_\pi^2]
\end{aligned}$$

Self energy – final result

Chosen (off-shell) renormalization: $\hat{\Pi}(0) = 0$

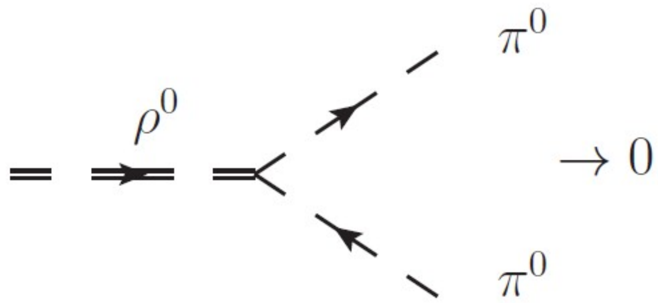
$$\begin{aligned}\hat{\Pi}(q^2) &= -g^2 \int_0^1 \mathrm{d}x \frac{1}{(4\pi)^2} \big(\log \Delta - \log(m_\pi^2) \big)(1-2x)^2 \\ &= -\frac{g^2}{(4\pi)^2} \left(-\frac{8}{9} + \frac{8}{3}\frac{m_\pi^2}{q^2} - \frac{2}{3}\sqrt{\left(\frac{4m_\pi^2 - q^2}{q^2}\right)^3} \arctan \sqrt{\frac{q^2}{4m_\pi^2 - q^2}} \right)\end{aligned}$$

Workflow

- Calculation of self energy --- Done
- Isospin factors (not done here)
- Calculation of Isospin=1 amplitude
- Partial-wave projection to P-wave
- Result: Scattering amplitude for the quantum numbers of the rho-meson:
  - Isospin *I*,
  - *G* parity,
  - total angular momentum *J*
    (=orbital angular momentum *L* for spinles particles)
  - Parity *P*
  - C-parity *C*

$$\rho(770) : \boxed{I^G(J^{PC}) = 1^+(1^{--})}$$

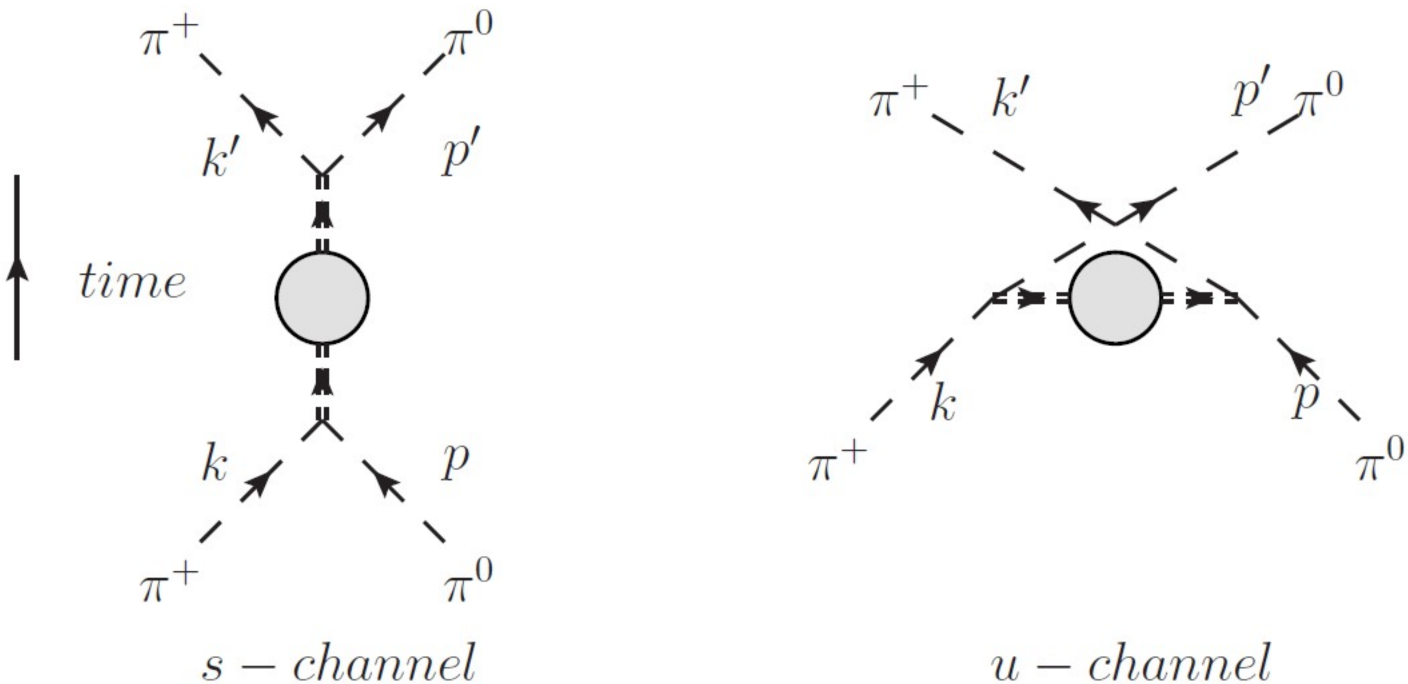
One remarks about isospin



Consider this process in the lab frame, the total angular momentum of ρ^0 is 1 since its spin is 1 and its space angular momentum is 0. On the other side, boson system $\pi^0\pi^0$ is symmetrical under exchange which requires the space momentum must be even: $l = 0, 2, 4\ldots$ $\big($parity transform $\boldsymbol{P}Y(r,\theta,\phi) = \boldsymbol{P}R(r)Y_L^m(\theta,\phi) = (-1)^L R(r)Y_L^m(\theta,\phi)\big)$. However, the spin of π is 0, the total angular momentum of $\pi^0\pi^0$ system cannot equal 1. Therefore, $\rho^0\pi^0\pi^0$ is 0.

Isospin eigenstates

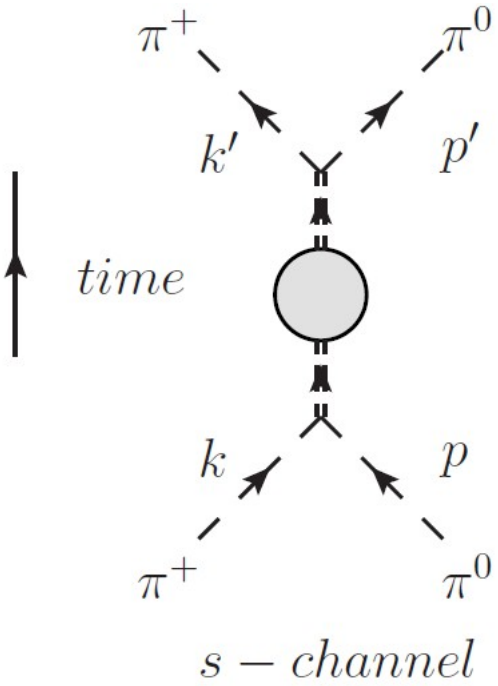
$$
\begin{aligned}
T^{I=1} &= \langle I=1, I_3=1|T|I=1, I_3=1\rangle \\
&= \frac{1}{2}\langle\pi^+\pi^0|T|\pi^+\pi^0\rangle - \frac{1}{2}\langle\pi^0\pi^+|T|\pi^+\pi^0\rangle - \frac{1}{2}\langle\pi^+\pi^0|T|\pi^0\pi^+\rangle + \frac{1}{2}\langle\pi^0\pi^+|T|\pi^0\pi^+\rangle
\end{aligned}
$$



S-channel rho-exchange

$$
\begin{aligned}
i\mathcal{M}_s &= (-ig)(k-p)_\mu \left[\frac{-i(g^{\mu\nu} - q^\mu q^\nu/q^2)}{q^2 - m_\rho^2} \frac{1}{1-\Pi'(q^2)} + \frac{iq^\mu q^\nu (1/m_\rho^2 - 1/q^2)}{q^2 - m_\rho^2} \right] (-ig)(k'-p')_\nu \\
&= -ig^2 \left[\frac{-(k-p)\cdot(k'-p') + \frac{(k^2-p^2)(k'^2-p'^2)}{q^2}}{(q^2-m_\rho^2)\big(1-\Pi'(q^2)\big)} + \frac{(k^2-p^2)(k'^2-p'^2)(1/m_\rho^2 - 1/q^2)}{q^2-m_\rho^2} \right] \\
&= ig^2 \frac{k\cdot k' + p\cdot p' - k\cdot p' - p\cdot k'}{(q^2-m_\rho^2)\big(1-\Pi'(q^2)\big)} \\
&= ig^2 \frac{\frac{u}{2} - m_\pi^2 + \frac{u}{2} - m_\pi^2 - \frac{t}{2} + m_\pi^2 - \frac{t}{2} + m_\pi^2}{(s-m_\rho^2)\big(1-\Pi'(s)\big)} \\
&= ig^2 \frac{u-t}{(s-m_\rho^2)\left(1 - \frac{s}{s-m_\rho^2}\Pi(s)\right)} \\
&= ig^2 \frac{u-t}{s - m_\rho^2 - s\Pi(s)}
\end{aligned}
$$

where $q = k + p = k' + p' = \sqrt{s}$ for s-channel.



Ignore u-channel rho-exchange:
give up crossing symmetry in favor of unitarity!

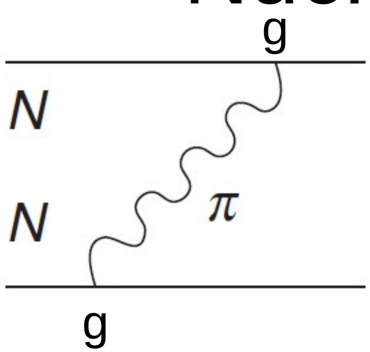
Pion-pion scattering with rho:
Messages to take home

- General workflow:
  - write down Lagrangian, derive Feynman rules
  - Construct a (s-channel) unitary amplitude by first calculating the self energy and then resum it.
  - As the rho field was included by minimal substitution into the pion kinetic term, $\rho\pi\pi$ AND $\rho\rho\pi\pi$ vertices appear → An additional tadpole self-energy diagram appears, needed to provide the correct Lorentz structure!
  - Project to isospin and total angular momentum
  - Fit free constants (coupling constant and bare mass) to experimental phase shifts
  - Extract poles (=resonance mass and width) and residues (= branching ratio) from the pole on the second Riemann sheet.

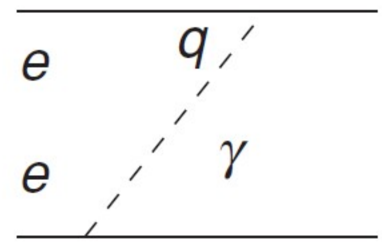
(continued)

- Poles are on the second (hidden) Riemann sheet. The first sheet is free of poles (see: Analyticity from Causality).
- The amplitude we constructed contained the rho in the s-channel and *u*-channel. To maintain *s*-channel unitarity, the *u*-channel term needs to be neglected (see: Mandelstam plane). *u*-channel unitarity violated. Crossing symmetry violated.
- More sophisticated schemes exist to simultaneously guarantee unitarity and crossing symmetry (Roy-Steiner equations).

Nuclear effective force – finite range



in analogy to



$$= \frac{e^2}{q^2}(\bar{u}\gamma^\mu u)(\bar{u}\gamma_\mu u).$$

$$D_\pi(q) = \frac{1}{\mu^2 - q^2} \qquad 1/q^2$$

→ Scattering amplitude: $A = \frac{g^2}{\mu^2 - q^2}$

What is this in the non-relativistic picture?

$$f = -\frac{2m}{4\pi}\int e^{i\mathbf{k}'\cdot\mathbf{r}} V(r)\,\psi(\mathbf{r})\,d^3r \quad \xrightarrow{\text{Born approximation}} \quad f_B = -\frac{2m}{4\pi}\int e^{i\mathbf{q}\cdot\mathbf{r}} V(r)\,d^3r$$

$\mathbf{q}$ is the momentum transfer, $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ $\qquad E = \mathbf{k}^2/2m$

$$|q_0| \sim \mathbf{q}^2/m \ll |\mathbf{q}|, \qquad \text{so that} \quad q^2 = q_0^2 - \mathbf{q}^2 \simeq -\mathbf{q}^2. \longrightarrow A \simeq \frac{g^2}{\mu^2 + \mathbf{q}^2}$$

position space: $V(r) = -\frac{4\pi}{2m}\int e^{-i\mathbf{q}\mathbf{r}} \frac{g^2}{\mu^2 + \mathbf{q}^2}\frac{d^3q}{(2\pi)^3} = \frac{g^2}{2m}\cdot\frac{e^{-\mu r}}{r}$

→ Effective interaction characterized by a finite radius $r_0 = 1/\mu$ EVEN with point-like interactions

$$d\sigma(a \to b) \equiv \frac{1}{J} |\mathcal{M}_{ab}|^2 (2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_{j \in b} k_j \right) \cdot \frac{1}{[n!]} \prod_{j \in b} d\Gamma(k_j)$$

$$d\Gamma_j = \left(\frac{1}{\sqrt{2k_{0j}}} \right)^2 \frac{d^3\mathbf{k}_j}{(2\pi)^3} = \frac{d^3\mathbf{k}_j}{2(2\pi)^3 k_{0j}} = \frac{d^4 k_j}{(2\pi)^4} \cdot 2\pi\delta_+ (k_j^2 - m_j^2)$$

- Lorentz invariant flux $J = 4p_c(s)\sqrt{s}$
- 2 particles incoming, $\mathbf{p}_1 = -\mathbf{p}_2 = (0, 0, p_c)$

$$p_c = p_c(s) = \frac{\sqrt{(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)}}{2\sqrt{s}}$$

- Eliminate multiple counting of physically indistinguishable configuration produced by permutation of identical particles; If in the final state there are n_s particles of type *s*,

$$\frac{1}{[n!]} \equiv \prod_s \frac{1}{n_s!}, \quad \sum_s n_s = n$$

- This slide is merely for your information and not derived in detail. More details: Gribov, Sec. 1.7, Peskin Schroeder Sec. 4.5