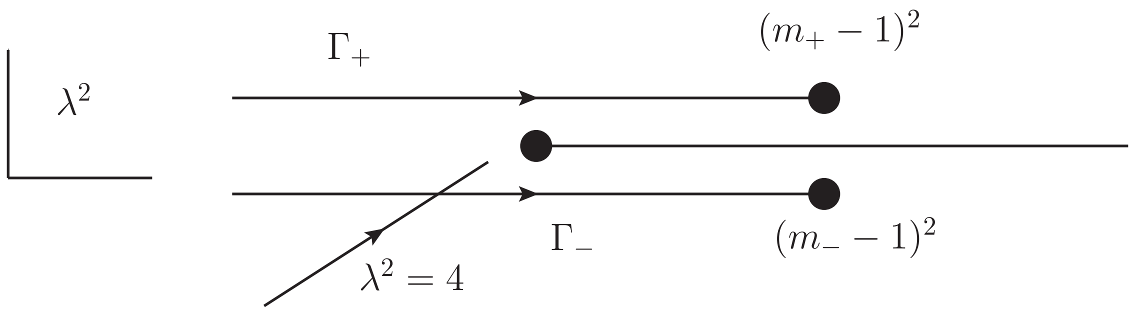
4. Three-body properties

$$\Phi(s,m^2) = M(s) +$$
$$2M(s)\int_{-\infty}^{(m-1)^2} \Delta_1(\lambda^2,m^2,s)\Phi(\lambda^2,m^2)$$

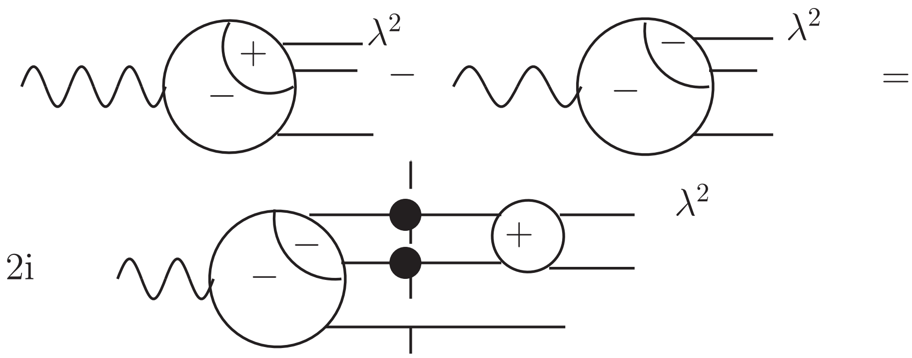
Interested in m^2 behaviour

4.1 Three-body unitarity



$$\boxed{\text{disc}_{m^2=9}\Phi(s_+,m^2)} = \Phi(s_+,m_+^2) - \Phi(s_+,m_-^2) =$$
$$2M(s_+)\int_{-\infty}^{(m-1)^2} \Delta_1[\Phi(\lambda_+^2,m_+^2)] - \Phi(\lambda_-^2,m_-^2)]$$
$$\Phi_{++} - \Phi_{--} = (\Phi_{++} - \Phi_{+-}) + (\Phi_{+-} - \Phi_{--})$$
$$= \boxed{\text{disc}_{m^2=9}\Phi(\lambda_+^2,m^2)} + \text{disc}_{\lambda^2=4}\Phi(\lambda^2,m_-^2)$$

Will get integral eqn for $\text{disc}_{m^2=9}\Phi$
We know $\text{disc}_{\lambda^2=4}\Phi(\lambda^2,m_-^2)$ from subenergy unitarity



$$F(\lambda^2, t(\lambda^2, x), m^2)$$

$$\text{disc}_{\lambda^2=4}\Phi(\lambda^2, m_-^2) =$$
$$2\text{i}\rho(\lambda_+^2)M(\lambda_+^2)F^0(\lambda_-^2, m_-^2)\theta(\lambda^2 - 4)$$

$$F^0(\lambda^2, m^2) = \tfrac{1}{2}\int_{-1}^{1} F(\lambda^2, t(\lambda^2, x), m^2)\text{d}x$$

Integral eqn: $\text{disc}_{m^2=9}\Phi(s_+, m^2) =$

$2M(s_+)\int_4^{(m-1)^2} \text{d}\lambda^2 \Delta_1 2\text{i}\rho(\lambda_+^2)M(\lambda^2)F^0(\lambda_-^2, m_-^2)$

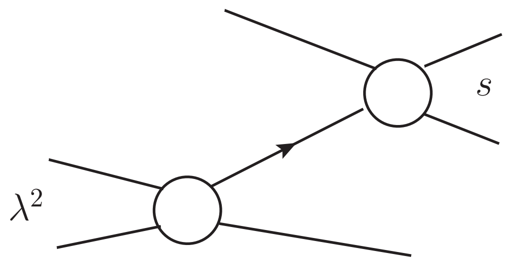
$+2M(s_+)\int_{-\infty}^{(m-1)^2} \text{d}\lambda^2 \Delta_1 \text{disc}_{m^2=9}\Phi(\lambda^2, m^2)$

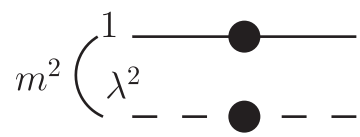
Solve by iteration

First iterate is the inhomogeneous term:

$$2M(s_+)\int_4^{(m-1)^2} \mathrm{d}\lambda^2 \Delta_1 2\mathrm{i}\rho(\lambda_+^2) M(\lambda^2) F^0(\lambda_-^2, m_-^2)$$

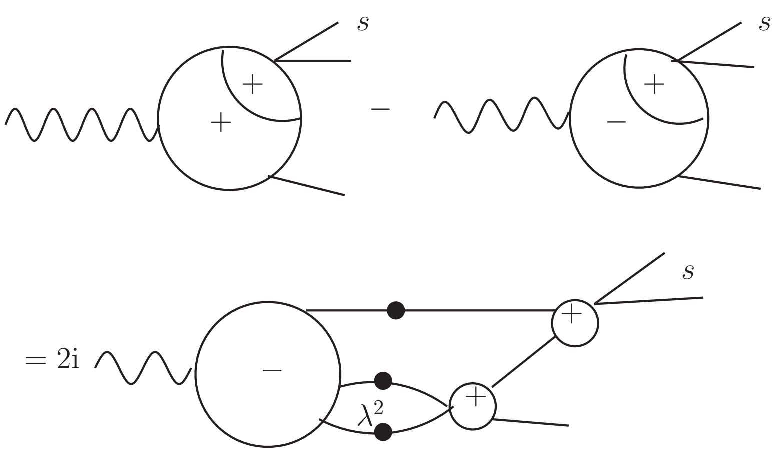
$$M(\lambda^2)\frac{\Delta_1(\lambda^2,m^2,s)}{\sigma(\lambda^2,m^2)}M(s) \equiv \Psi^{(1)}(\lambda^2,m^2,s) \equiv B(\lambda^2,m^2,s)$$

$J = 0$ 

$\sigma(\lambda^2, m^2)$ 

So first iteration gives

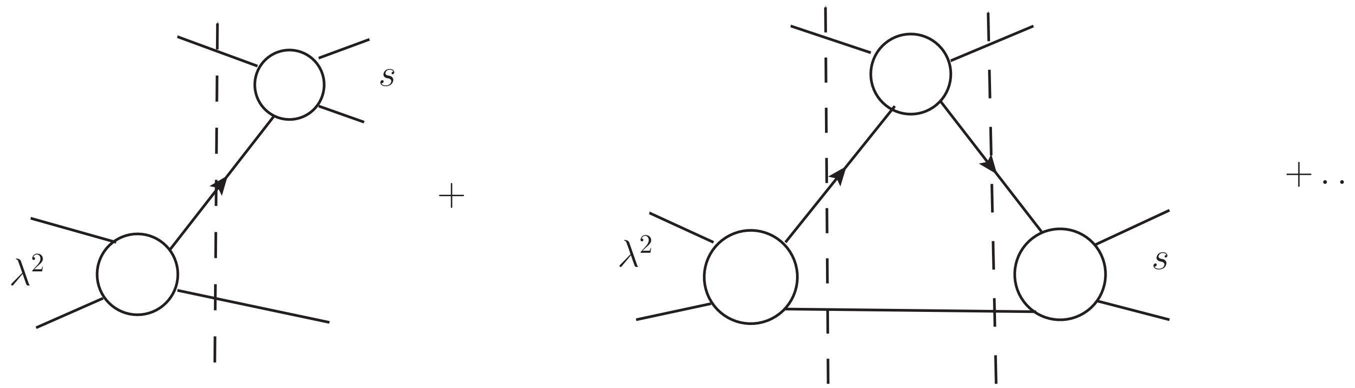
$$\mathrm{disc}^{(1)}_{m^2=9}\Phi(s_+, m^2) = 4\mathrm{i}\int_4^{(m-1)^2} \mathrm{d}\lambda^2 \{$$
$$F^0(\lambda_-^2, m_-^2)\rho(\lambda_+^2)\sigma(\lambda_+^2, m_+^2)\Psi^{(1)}(\lambda_+^2, m_+^2, s_+)\}$$



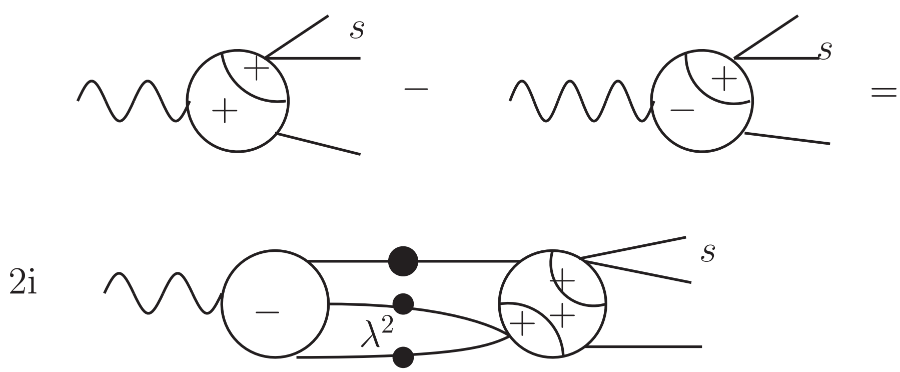
Note: phase space $= \frac{\mathrm{d}\lambda^2 \mathrm{d}t}{m^2} = \sigma(\lambda^2, m^2)\rho(\lambda^2)\mathrm{d}\lambda^2 \mathrm{d}x$

All orders: $\Psi^{(1)}(\lambda^2, m^2, s) \to \Psi(\lambda^2, m^2, s)$ where

$$\Psi(\lambda^2, m^2, s) = B(\lambda^2, m^2, s) + 2M(s) \int_{-\infty}^{(m-1)^2} \mathrm{d}\mu^2 \Delta_1(\mu^2, m^2, s) \Psi(\lambda^2, m^2, \mu^2)$$



$$\mathrm{disc}_{m^2=9} \Phi(s_+, m^2) = 4\mathrm{i} \int_4^{(m-1)^2} \mathrm{d}\lambda^2 \{ F^0(\lambda_-^2, m_-^2) \rho(\lambda_+^2) \sigma(\lambda_+^2, m_+^2) \Psi(\lambda_+^2, m_+^2, s_+) \}$$



The model satisfies THREE body unitarity!

But note:

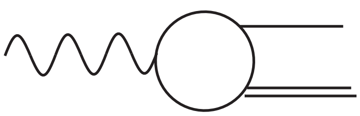
(a) Ψ should be symmetric in $s \leftrightarrow \lambda^2$; not the case for $\lambda^2 \leq 0$ bits $\rightarrow$ cut-off at $\lambda^2 = 0$

(b) "rescattering series" not same as Feynman graphs (apart from triangle)

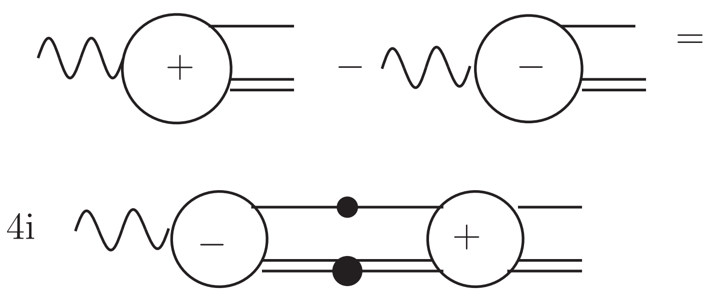
In practice, especially within the context of an isobar-like approach, we don't focus on three-body unitarity, but rather on "quasi two-body" unitarity i.e. particle + resonance scattering

4.2 Particle-resonance scattering

$\Phi(s, m^2) = M(s)\phi(s, m^2)$: $M(s)$ has $s \geq 4$ cut + pole in sheet **II** at $s^{\mathrm{II}} = s_{\mathsf{C}}$. $\phi(s, m^2)$ has same cut, but not the second sheet pole. Define particle-resonance production amplitude by $\phi(s_{\mathsf{C}}, m^2) = \lim_{s^{\mathrm{II}} \to s_{\mathsf{C}}} \frac{(s_{\mathsf{C}} - s^{\mathrm{II}})}{g^2} \Phi(s^{\mathrm{II}}, m^2)$

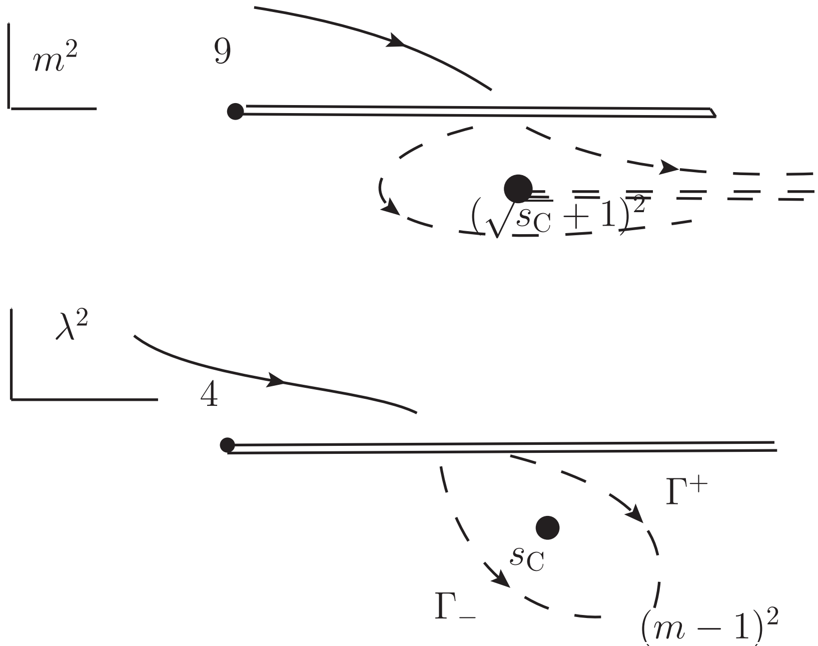


we are interested in m^2 structure of $\phi(s_{\mathsf{C}}, m^2)$, specifically identifying particle + resonance branch point at $m^2 = (\sqrt{s_{\mathsf{C}}} + 1)^2$ and associated disc



$$\Phi(s,m^2) = M(s) +$$
$$2\int^{(m-1)^2} \mathrm{d}\lambda^2 \Delta_1(\lambda^2, m^2, s) M(\lambda^2)\phi(\lambda^2, m^2)$$

1. Branch point at $m^2 = (\sqrt{s_C}+1)^2$



As we continue down in m^2 through the $m^2 \geq 9$ cut, the end-point of the λ^2-contour at $\lambda^2 = (m-1)^2$ goes into the second λ^2 sheet across the $\lambda^2 \geq 4$ cut, and will hit the pole at $\lambda^{2\mathrm{II}} = s_C$. Singularity at $(m-1)^2 = s_C$ i.e. $m^2 = (\sqrt{s_C}+1)^2$. Will see it is sq root branch point ("woolly cut")

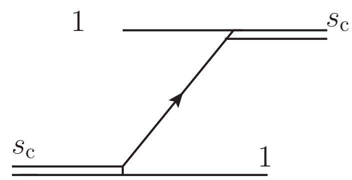
2. Discontinuity across woolly cut = difference between two m^2-continuations which leave the pole on RHS of λ^2 contour and on LHS of λ^2 contour. This difference is

$\Phi(s_+, m_+^2) - \Phi(s_+, m_-^2) = 2M(s_+) \int^{(m-1)^2} \{$
$d\lambda^2 \Delta_1 [M(\lambda_+^{2\mathrm{II}})\phi(\lambda^{2\mathrm{II}}, m_+^2) - M(\lambda_-^{2\mathrm{II}})\phi(\lambda^{2\mathrm{II}}, m_-^2)]\}$
$[\ldots] = M(\lambda_+^{2\mathrm{II}})[\phi(\lambda^{2\mathrm{II}}, m_+^2) - \phi(\lambda^{2\mathrm{II}}, m_-^2)]$
$+\phi(\lambda^{2\mathrm{II}}, m_-^2)[M(\lambda_+^{2\mathrm{II}}) - M(\lambda_-^{2\mathrm{II}})]$

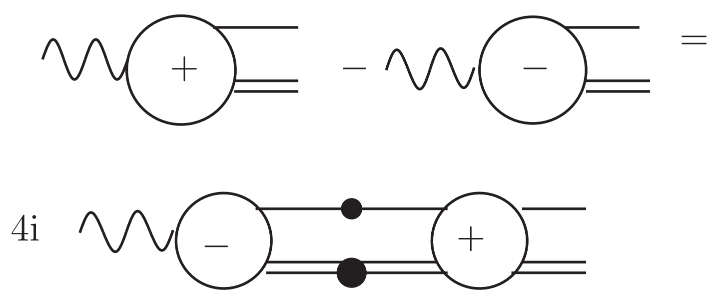
Last difference is $2\pi i \phi(\lambda^{2\mathrm{II}}, m_-^2)\delta(\lambda^{2\mathrm{II}} - s_{\mathrm{C}})$.
First difference is $\Phi(\lambda_+^{2\mathrm{II}}, m_+^2) - \Phi(\lambda_+^{2\mathrm{II}}, m_-^2)$

So $\mathrm{disc}_{wc}\Phi(s_+, m^2) =$
$4\pi i g^2 M(s_+)\phi(s_{\mathrm{C}}, m^2)\Delta_1(s_{\mathrm{C}}, m^2, s_+)+$
$2M(s_+) \int^{(m-1)^2} d\lambda^2 \Delta_1(\lambda^2, m^2, s_+)\mathrm{disc}_{wc}\Phi(\lambda_+^{2\mathrm{II}}, m^2)$
Another integral eqn for the disc, but this time the inhomogeneous term is not an integral!

First iteration is inhomogeneous term which is
$[\mathrm{disc}_{wc}\Phi(s_+, m^2)]^{(1)} =$
$4\pi \mathrm{i} g^2 M(s_+)\phi(s_\mathrm{C}, m_-^2)\Delta_1(s_\mathrm{C}, m^2, s_+)$
Keeping pole contribution on both sides,
$[\mathrm{disc}_{wc}\phi(s_\mathrm{C}, m^2)]^{(1)} = 4\pi \mathrm{i} g^2 \phi(s_\mathrm{C}, m_-^2)\Delta_1(s_\mathrm{C}, m^2, s_\mathrm{C})$
Now recall that Δ_1/σ is proportional to $J = 0$
projection of RPE process

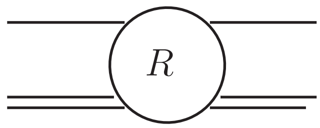


Carrying out iteration to all orders,
$[\mathrm{disc}_{wc}\phi(s_\mathrm{C}, m^2)] =$
$4\pi \mathrm{i}\phi(s_\mathrm{C}, m_-^2)\sigma(s_\mathrm{C}, m^2)R(s_\mathrm{C}, m_+^2, s_\mathrm{C})$

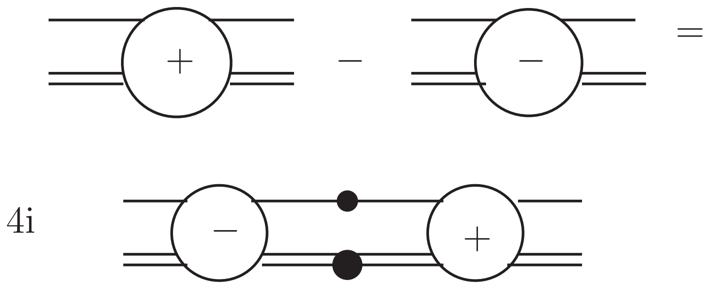


$\sigma(s_\mathrm{C}, m^2) =$
$\{[m^2 - (\sqrt{s_\mathrm{C}} + 1)^2][m^2 - (\sqrt{s_\mathrm{C}} - 1)^2]\}^{1/2}/m^2$

Here R is the particle-resonance scattering

amplitude 

R satisfies quasi two-body discontinuity



Substantial m^2-dependence can be generated in some cases, using truncated integration. Could impact the extraction of resonance pole positions.

Implementation of these woolly disc relations: use effective K matrix/P-vector formalism, with phase space σ instead of ρ

Summary

- two-body unitarity + analyticity + crossing $\rightarrow$ linear single-variable integral equations for isobar correction functions

- "minimal" set of constraints

- needs only two-body amplitudes

- employs standard angular momentum decomposition of IM

- surprising (?) three-body structure included

- now: include in exptal fits and compare with other approaches (effective Hamiltonians, relativistic scattering formalisms)