Three-Body Decay in the Isobar Model
Vincent Mathieu
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1 Summary

Let us consider the three-body decay

$$0(\Lambda, p_0) \to 1(\lambda_1, p_1) + 2(\lambda_2, p_2) + 3(\lambda_3, p_3). \tag{1}$$

m_i and J_i are the mass and spin of particle $i = 0, 1, 2, 3$. $\{i, j, k\}$ form a permutation of 123 and $\sigma_k = (p_i + p_j)^2$. The Mandelstam variables obey $\sigma_1 + \sigma_2 + \sigma_3 = m_0^2 + m_1^2 + m_2^2 + m_3^2$. In the center-of-momentum of particle 0 (CMF), the vector $\vec{p}_1^*$, $\vec{p}_2^*$ and $\vec{p}_3^*$ form a plane, the decay plane. The decay plane is the xz plane with the z axis being the opposite direction of $\vec{p}_1^*$ in the CMF and the y axis being the opposite direction of $\vec{p}_1^* \times \vec{p}_2^*$ in the CMF (to match with Misha's convention). All quantities in the CMF have a *. The CMF is also called the (123)RF. Notation:

- λ_i^* are the helicities of particle $i = 1, 2, 3$ in the (123) RF
- Λ^* is the spin projection of particle 0 along the z axis in the (123) RF
- λ_i are the helicities of particle $i = 0, 1, 2, 3$ in the isobar (ij) RF.
- Λ_k is the spin projection along $-\vec{p}_k$ of the isobar (ij) in its RF. It is also the helicity of the isobar in the (123) RF.
- s_k is the spin of the isobar (ij).

The amplitudes are written as

$$M_{\Lambda_{\text{Lab}},\{\lambda^*\}}(\sigma_1, \sigma_2) = \sum_{\Lambda^*} D^{J_0 *}_{\Lambda_{\text{Lab}},\Lambda^*}(\alpha, \beta, \gamma) O_{\Lambda^*,\{\lambda^*\}}(\sigma_1, \sigma_2) \tag{2a}$$

$$O_{\Lambda^*,\{\lambda^*\}}(\sigma_1, \sigma_2) = O^1_{\Lambda^*,\{\lambda^*\}}(\sigma_1, \sigma_2) + O^2_{\Lambda^*,\{\lambda^*\}}(\sigma_1, \sigma_2) + O^3_{\Lambda^*,\{\lambda^*\}}(\sigma_1, \sigma_2) \tag{2b}$$

$$O^k_{\Lambda^*,\{\lambda^*\}} = \sum_{s_k,\Lambda_k,\{\lambda\}} C^{s_k,\Lambda_k}_{\Lambda,\{\lambda^*\},\{\lambda\}} h^{0\to(ij)+k}_{\Lambda_k,\lambda_k^*}(m_0^2) h^{s_k\to i+j}_{\lambda_i,\lambda_j}(\sigma_k) \tag{2c}$$

The code returns the recoupling coefficients

$$C^{s_k,\Lambda_k}_{\Lambda,\{\lambda^*\},\{\lambda\}}(\sigma_1, \sigma_2) = d^{J_0}_{\Lambda^*,\Lambda_k-\lambda_k^*}(\theta^*_{k1}) d^{s_k}_{\Lambda_k,\lambda_i-\lambda_j}(\theta_{ij}) \times d^{J_i}_{\lambda_i,\lambda_i^*}(-\omega_{i|k}) d^{J_j}_{\lambda_j,\lambda_j^*}(\omega_{j|k}) \tag{3}$$

2 Details

The decaying particle (0) might be polarized and that polarization depends in the particular experiment. One can either choose to rotate the system, for each event, such that the spin density matrix of particle 0 takes a simple for or either use an event-dependent spin density matrix. In the former case, the amplitudes in the lab frame read

$$M_{\Lambda_{\mathrm{Lab}},\{\lambda^*\}}(\sigma_1,\sigma_2) = \sum_{\Lambda^*} D^{J_0*}_{\Lambda_{\mathrm{Lab}},\Lambda^*}(\alpha,\beta,\gamma) O_{\Lambda^*,\{\lambda^*\}}(\sigma_1,\sigma_2). \tag{4}$$

And in both cases the intensity reads (ignoring an overall factor), with P_0 the polarization and Ω_P the angles of the polarization vector

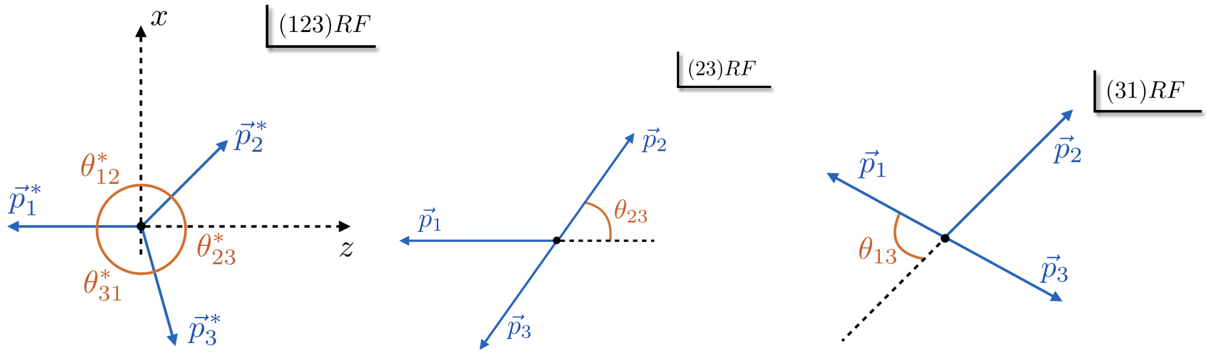
$$I(P_0,\Omega_P,\sigma_1,\sigma_2) = \sum_{\Lambda^*,\bar{\Lambda}^*,\{\lambda^*\}} \bar{\rho}_{\Lambda^*,\bar{\Lambda}^*}(P_0,\Omega_P) O_{\Lambda^*,\{\lambda^*\}}(\sigma_1,\sigma_2) O^*_{\bar{\Lambda}^*,\{\lambda^*\}}(\sigma_1,\sigma_2). \tag{5}$$

The interesting function $O_{\Lambda^*,\{\lambda^*\}}(\sigma_1,\sigma_2)$ is called the Dalitz plot function and can be parametrized in terms of isobars.

In the isobar model, one decomposes the Dalitz function into three decay chains

$$O_{\Lambda^*,\{\lambda^*\}}(\sigma_1,\sigma_2) = O^1_{\Lambda^*,\{\lambda^*\}}(\sigma_1,\sigma_2) + O^2_{\Lambda^*,\{\lambda^*\}}(\sigma_1,\sigma_2) + O^3_{\Lambda^*,\{\lambda^*\}}(\sigma_1,\sigma_2). \tag{6}$$

Let us consider the decay chain $0 \to (ij) + k$ followed by $(ij) \to i + j$ with i,j,k being a permutation of $1,2,3$. We write the isobar decay in the isobar rest frame, *i.e.* the (ij) RF for the isobar k. Its spin projection Λ_k is quantized along the opposite direction of $\vec{p}_k^*$. The (ij) RF is related to the CMF by a boost along $\vec{p}_k^*$, which leaves Λ_k invariant but Wigner-rotates the helicities λ_i and λ_j in the (ij) RF.



The isobar decay is written in the (ij) RF

$$h^{s_k \to i+j}_{\lambda_i,\lambda_j}(\sigma_k) d^{s_k}_{\Lambda_k,\lambda_i-\lambda_j}(\theta_{ij}) \tag{7}$$

The scattering angle θ_{ij} is the angle between $\vec{p}_i$ and $-\vec{p}_k$ in the (ij) rest frame (with $\theta_{ji} = \pi - \theta_{ij}$):

$$\cos\theta_{ij} = -\frac{\sigma_k(\sigma_i - \sigma_j) + (m_i^2 - m_j^2)(m_0^2 - m_k^2)}{\lambda^{1/2}(\sigma_k, m_i^2, m_j^2)\lambda^{1/2}(m_0^2, m_k^2, \sigma_k)} \tag{8}$$

This isobar decay amplitude needs to be boosted to the CMF. This boost introduces the Wigner rotations:

$$d^{J_i}_{\lambda_i,\lambda_i^*}(-\omega_{i|k}) d^{J_j}_{\lambda_j,\lambda_j^*}(\omega_{j|k}) = (-1)^{\lambda_i-\lambda_i^*} d^{J_i}_{\lambda_i,\lambda_i^*}(\omega_{i|k}) d^{J_j}_{\lambda_j,\lambda_j^*}(\omega_{j|k}) \tag{9}$$

$\omega_{i|k}$ is the angle of the Wigner rotation bringing particle i from the isobar k rest frame to the CMF. On the next page, it is shown that $\omega_{j|k}$ is positive when ijk are even permutation of 123. Since the angles are specified by their cosine, we need to take the sign into account. Eq. (9) assumes that this is the case, otherwise the sign are on the other Wigner-d. The angle Wigner rotation bringing particle i from the $(ij) = k$ rest frame to the overall rest frame (123) RF is

$$\cos\omega_{i|k} = \frac{(\sigma_k + m_i^2 - m_j^2)(m_0^2 + m_i^2 - \sigma_i) - 2m_i^2(\sigma_k - m_k^2 + m_0^2)}{\lambda^{1/2}(\sigma_k, m_i^2, m_j^2)\lambda^{1/2}(m_0^2, m_i^2, \sigma_i)} \tag{10}$$

$\omega_{i|k}$ is the angle of the rotation bringing the direction of $\vec{p}_i + \vec{p}_j|_{(i)RF}$ to the direction of $\vec{p}_1 + \vec{p}_2 + \vec{p}_3|_{(i)RF}$, that is, the angle between the boosting direction of particle i from its RF to (ij) RF and the boosting direction of particle i from its RF to the CMF.

Finally particle 0 decay into the isobar $k = (ij)$ and particle k yields

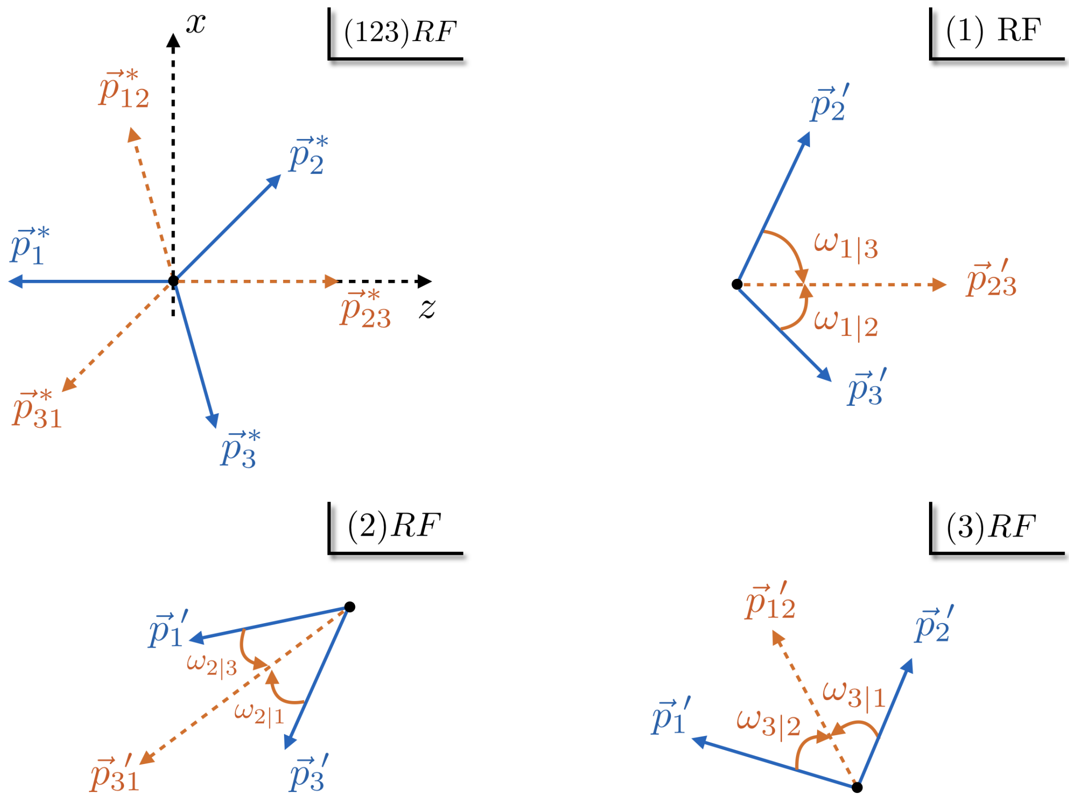
$$h^{0\to(ij)+k}_{\Lambda_k,\lambda_k^*}(m_0^2) d^{J_0}_{\Lambda^*,\Lambda_k-\lambda_k^*}(\theta^*_{k1}). \tag{11}$$

The angle θ^*_{k1} between the isobar k $(-\vec{p_k^*})$ and the z axis $(-\vec{p_1^*})$ is given by the angles θ^*_{ij} between $\vec{p_i^*}$ and $\vec{p_j^*}$ in the (123) rest frame. Obviously $\theta^*_{ij} = \theta^*_{ji}$, $\theta^*_{kk} = 0$ and $\theta^*_{12} + \theta^*_{23} + \theta^*_{31} = 2\pi$:

$$\cos\theta^*_{ij} = \frac{(m_0^2 + m_i^2 - \sigma_i)(m_0^2 + m_j^2 - \sigma_j) + 2m_0^2(m_i^2 + m_j^2 - \sigma_k)}{\lambda^{1/2}(m_0^2, m_i^2, \sigma_i)\lambda^{1/2}(m_0^2, m_j^2, \sigma_j)} \tag{12}$$

Collecting all the pieces we obtain

$$\begin{aligned} O^k_{\Lambda^*,\{\lambda^*\}} = \sum_{s_k,\Lambda_k,\{\lambda\}} & h^{0\to(ij)+k}_{\Lambda_k,\lambda_k^*}(m_0^2) d^{J_0}_{\Lambda^*,\Lambda_k-\lambda_k^*}(\theta^*_{k1}) \\ & \times h^{s_k\to i+j}_{\lambda_i,\lambda_j}(\sigma_k) d^{s_k}_{\Lambda_k,\lambda_i-\lambda_j}(\theta_{ij}) \times d^{J_i}_{\lambda_i,\lambda_i^*}(-\omega_{i|k}) d^{J_j}_{\lambda_j,\lambda_j^*}(\omega_{j|k}) \end{aligned} \tag{13}$$



To compute the angle of the Wigner rotation, I use a covariant represenation of its cosine with two 4-vectors a and b. They are such that before the boost, a is in its rest frame, *i.e* $\boldsymbol{a} = \mathbf{0}$ and after the boost b is in its rest frame, *i.e* $\Lambda \boldsymbol{b} = \mathbf{0}$. We can then express the initial and final energies as $E = a \cdot p$ and $E' = (\Lambda b) \cdot p' = b \cdot p$ (up to irrelevant normalizations that will cancel in $\cos \omega$). We obtain the Wigner angle for boost (no rotation) in term of the vectors (a, b) describing the boost in the following covariant form [Gomatam:1974za]

$$\cos \bar{\omega} = \frac{(p \cdot a)(p \cdot b) - m^2 a \cdot b}{\{[(p \cdot a)^2 - m^2 a^2][(p \cdot b)^2 - m^2 b^2]\}^{1/2}}. \tag{14}$$

It shows that the Wigner angle corresponds to the angle between the vectors a and b in p rest frame !

In our case, we want to boost from the (ij) RF, so $a = p_i + p_j$ to the CMF so $b = p_i + p_j + p_k$. In the (i) RF, these directions correspond to the directions of $\vec{p_j}'$ and $\vec{p_j}' + \vec{p_k}'$. The prime denoting vectors in a particle rest frame.

In the figure, let's consider the isobar (23) ($k = 1$). The Wigner angle $\omega_{2|1}$ for particle 2 is the direction between $\vec{p_3}'$ and $\vec{p_3}' + \vec{p_1}' \equiv \vec{p_{31}}'$. The Wigner angle $\omega_{3|1}$ for particle 3 is the direction between $\vec{p_2}'$ and $\vec{p_2}' + \vec{p_1}' \equiv \vec{p_{12}}'$ because in the (3)RF, the spin projection of particle 3 is quantized along the opposite direction of $\vec{p_2}'$ and in the CMF its helicity is along the opposite direction of $\vec{p}_{12}^{\,*}$ so one needs to rotate from $\vec{p_2}'$ to $\vec{p_{12}}'$.

On the figure, we can indeed see an opposite sign between the two angles $\omega_{2|1}$ and $\omega_{3|1}$. We also see that that for a given particle i, its Wigner angles $\omega_{i|j}$ and $\omega_{i|k}$ of the boosting from the j-decay chain and the k-decay chain respectively are opposite. Also the two Wigner rotation $\omega_{i|k}$ and $\omega_{j|k}$ needed in the recoupling coefficient in the k-chain have opposite sign. Moreover $\omega_{j|k}$ is positive if ijk is an even permutation of 123 if the CMF momenta $\vec{p}_1^{\,*}$, $\vec{p}_2^{\,*}$ and $\vec{p}_3^{\,*}$ are labelled in the positive sense (anti-clockwise).